PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 84, Number 1, January 1982

A STABILITY PROPERTY OF A CLASS OF BANACH SPACES NOT CONTAINING A COMPLEMENTED COPY OF l_1

ELIAS SAAB AND PAULETTE SAAB

ABSTRACT. Let E be a Banach space and K be a compact Hausdorff space. The space $C(K, E)$ will stand for the Banach space of all continuous E-valued functions on K equipped with the sup norm. It is shown that the space E does not contain a complemented subspace isomorphic to l_1 if and only if $C(K, E)$ has the same property.

Let E be a Banach space, let $(\Omega, \Sigma, \lambda)$ be a finite measure space. The classical Banach spaces l_1, c_0, $L_p(\lambda)$ and $L_p(\lambda, E)$ will have their usual meaning [3]. The notations and terminology used and not defined in this paper can be found in [3], [4], [7].

In [5] Kwapien showed that if c_0 embeds in $L_p(\lambda, E)$ then c_0 embeds into E if $1 \leqslant p < +\infty$. Pisier [6] showed that if l_1 embeds into $L_p(\lambda, E)$ then l_1 embeds in E if $1 < p < +\infty$. The moral behind Kwapien's theorem is: Since c_0 cannot embed into $L_p(\lambda)$ if $1 \leqslant p < +\infty$, c_0 must embed into E if it does embed in $L_p(\lambda, E)$. A similar remark can be made about Pisier's result.

In this paper we will show that if l_1 is isomorphic to a complemented subspace of $C(K, E)$, then l_1 is isomorphic to a complemented subspace of E and the moral behind our result is that l_1 is not isomorphic to a complemented subspace of any $C(K)$ space.

THEOREM 1. *Let K be a compact Hausdorff space and E be a Banach space; then l_1 is isomorphic to a complemented subspace of $C(K, E)$ if and only if l_1 is isomorphic to a complemented subspace of E.*

PROOF. If l_1 is isomorphic to a complemented subspace of $C(K, E)$, then c_0 embeds in $C(K, E)^*$ [1]. The space $C(K, E)^*$ is isometrically isomorphic to the Banach space $M(K, E^*)$ of all w^*-regular E^*-valued measures of bounded variation defined on the σ-field Σ of Borel subsets of K and equipped with the norm $\|m\| = |m|(K)$, where $|m|$ is the variation of m. Let $(m_n)_{n \geq 1}$ be a sequence in $M(K, E^*)$ equivalent to the usual c_0-basis and let λ be the scalar measure defined on Σ by $\lambda = \sum_{n=1}^{\infty} |m_n|/2^n$. Let Σ_1 be the completion of Σ with respect to λ.

Received by the editors December 20, 1980.
1980 *Mathematics Subject Classification.* Primary 46G10, 46B22.
Key words and phrases. Complemented subspaces, vector measures.

0002-9939/82/0000-0010/$01.75

Fix ρ a lifting of $\mathcal{L}^\infty(\Sigma_1, \lambda)$ [4, §11]. For each $n \geq 1$, there exists a function g_n: $K \to E^*$ such that

(i) For every $x \in E$, the map $t \to \langle g_n(t), x\rangle$ is λ-integrable.

(ii) For every $A \in \Sigma_1$ and every $x \in E$

$$\langle m_n(A), x\rangle = \int_A \langle g_n(t), x\rangle \, d\lambda.$$

(iii) $\rho(g_n) = g_n$ (see [4, p. 212]).

(iv) The map $t \to \|g_n(t)\|$ is λ-integrable and $\|m_n\| = \int_K \|g_n(t)\| \, d\lambda$.

The existence of each g_n satisfying (i)–(iv) is assured by [4, §11, Theorem 5]. Let $(a_n)_{1 \leq n \leq p}$ be a finite real sequence and let $m = \sum_{n=1}^p a_n m_n$. By [4, §11, Theorem 5], there exists a function g: $K \to E^*$ satisfying with respect to m and λ the above properties (i)–(iv); therefore $\rho(g) = g$ and $\|m\| = \int_K \|g(t)\| \, d\lambda$, and for every $x \in E$ and every A in Σ_1

$$(*) \qquad \langle m(A), x\rangle = \int_A \langle g(t), x\rangle \, d\lambda.$$

Let $h = \sum_{n=1}^p a_n g_n$; the properties (ii) and $(*)$ imply that for every $x \in E$

$$(**) \qquad \langle h(t), x\rangle = \langle g(t), x\rangle, \quad \lambda\text{-almost everywhere.}$$

The properties $(**)$, $\rho(g) = g$ and $\rho(h) = h$ imply that $g = h$ [4, p. 212]. Hence

$$\left\| \sum_{n=1}^{p} a_n m_n \right\| = \int_K \left\| \sum a_n g_n(t) \right\| d\lambda$$

for any finite sequence of reals $(a_n)_{1 \leq n \leq p}$. Let F be the space of all finite real sequences and denote by e_n the nth unit vector. For $a = (a_n)_n \in F$, let $\|a\|_\infty = \sup_n |a_n|$. For each $t \in K$ define a seminorm $|\ |_t$ on F by $|a|_t = \|\sum a_n g_n(t)\|$. Clearly $|a| = \int_K |a|_t \, d\lambda$ is a seminorm of F and $|a| = \|\sum a_n m_n\|$. Since (m_n) is equivalent to the c_0-basis in $M(K, E^*)$ we have

$$C_2 \|a\|_\infty \leq \left\| \sum a_n m_n \right\| \leq C_1 \|a\|_\infty$$

for some $C_1 > 0$ and some $C_2 > 0$, but this implies that $C_2\|a\|_\infty \leq |a| \leq C_1\|a\|_\infty$ and this means that $(e_n)_{n \geq 1}$ is equivalent to the c_0 basis for $|\ |$ in F. By [2, Theorem 1], there exist $t \in K$ and a subsequence of $(e_n)_{n \geq 1}$ which is a c_0-basis for $|\ |_t$. Hence there exists a subsequence (g_{n_p}) of (g_n) such that $(g_{n_p}(t))$ is equivalent to the usual c_0-basis in E^*; therefore c_0 embeds in E^*. Consequently l_1 is isomorphic to a complemented subspace of E by [1]. The other implication is of course obvious.

Suppose now that K is a compact convex subset of a locally convex Hausdorff topological vector space and let $A(K, E)$ stand for the Banach space of all affine E-valued continuous functions equipped with the supremum norm. Theorem 1 and Theorem 3.4 of [7] give a more general result, namely,

COROLLARY 2. *Suppose that K is a Choquet simplex. Then l_1 is isomorphic to a complemented subspace of $A(K, E)$ if and only if l_1 is isomorphic to a complemented subspace of E.*

PROOF. Note that since K is a Choquet simplex, the dual of $A(K, E)$ is isometrically isomorphic to a closed subspace of $M(K, E^*)$ [7, Theorem 3.4]. By [1], c_0 embeds in $A(K, E)^*$ and therefore c_0 embeds in $M(K, E^*)$. Hence c_0 embeds in E^* by Theorem 1 and consequently l_1 is isomorphic to a complemented subspace of E.

BIBLIOGRAPHY

1. C. Bessaga and A. Pełczynski, *On bases and unconditional convergence of series in Banach spaces*, Studia Math. **17** (1958), 151–164.

2. J. Bourgain, *An averaging result for c_0-sequences*, Bull. Soc. Math. Belg. Ser. B **30** (1978), 83–87.

3. J. Diestel and J. J. Uhl, *Vector measures*, Math. Surveys, no. 15, Amer. Math. Soc., Providence, R.I., 1977.

4. N. Dinculeanu, *Vector measures*, Pergamon Press, New York, 1967.

5. S. Kwapien, *Sur les espaces de Banach contenant c_0*, Studia Math. **52** (1974), 187–188.

6. G. Pisier, *Une propriété de stabilité de la classe des espaces ne contenant pas l_1*, C. R. Acad. Sci. Paris Sér. A **286** (1978), 747–749.

7. P. Saab, *The Choquet integral representation in the affine vector valued case*, Aequationes Math. **20** (1980), 252–262.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BRITISH COLUMBIA, VANCOUVER, BRITISH COLUMBIA, CANADA

Current address: Department of Mathematics, University of Missouri-Columbia, Columbia, Missouri 65211