PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 84, Number 1, January 1982

AN EXTREMAL VECTOR-VALUED L^p-FUNCTION TAKING NO EXTREMAL VECTORS AS VALUES

PETER GREIM

ABSTRACT. We give an example of a nonseparable Banach space V and a function x on $[0, 1]$ with values in the unit sphere of V that is an extreme point of the unit balls of all Bochner L^p-spaces $L^p(\lambda, V)$, $1 < p \leqslant \infty$, λ Lebesgue measure, though none of its values is an extreme point of the unit ball of V. This shows that a characterization of the extremal elements in $L^p(\lambda, V)$ for separable V, given by J. A. Johnson, does not hold in general.

The extremal elements in the unit sphere of vector-valued CK- or L^p-spaces have been studied by many authors, e.g. in [**2**, **5** and **6**]. (For the definition and elementary properties of Bochner L^p-spaces we refer the reader to [**4**].) A quite natural question is to ask whether such a function x is extremal if and only if

(1) the function $\|x(\cdot)\|$ is extremal in the corresponding scalar function space and

(2) the vector $x(t)$ is extremal in the ball with radius $\|x(t)\|$ for all t in a dense subset of the base space (in the CK-case) resp. for almost all t (in the L^p-case).

The "if" part is easy and well known; on the other hand, necessity of (1) is trivial. Hence the remaining question is the necessity of (2).

In the case $p = 1$ the necessity is easily seen, since (1) implies that the support of x is an atom (see also [**6**]).

The CK-case was settled long ago. Blumenthal, Lindenstrauss, and Phelps have shown in [**2**] that for real range spaces V with dimension $\leqslant 3$ the condition (2) is necessary. On the other hand they give an example of a 4-dimensional space V and an extremal x in $C([0, 1], V)$ taking no extremal values. In the remaining cases $1 < p \leqslant \infty$, J. A. Johnson [**5**] has shown the necessity of (2), provided V is *separable* and the measure is a Borel measure on a Polish space (see also [**6**]).

We shall give an example of a (nonseparable) Banach space W and a function f: $[0, 1] \to W$ that is extremal in the unit balls of all $L^p(\lambda, W)$ ($1 < p \leqslant \infty$, λ Lebesgue measure), although the function does not take any extremal values.

We start from a Banach space W and a function f: $[0, 1] \to W$, extremal in $C([0, 1], W)$, but taking no extremal values, that came up in a discussion with E. Behrends and R. Evans. Then we show that this example works also in the cases $L^p(\lambda, W)$, $1 < p \leqslant \infty$, using the representation $L^p(\lambda, W) \cong L^p(m, W)$ where m is a suitable measure on some Stonean space.

Received by the editors February 9, 1981 and, in revised form, April 27, 1981.
1980 *Mathematics Subject Classification*. Primary 46E40; Secondary 46B20.
Key words and phrases. Bochner L^p-space, extreme point, Stonean space.

Let K be the Stonean space of the measure algebra $\mathbf{B}/\lambda$ (Borel sets modulo λ-null sets) and m be the perfect Borel measure on K uniquely determined by $m(C) = \lambda(M)$, where the clopen subset C of K represents the equivalence class of M. (Here m is called *perfect* if each open set has positive measure and each nowhere dense set has measure zero. Each Borel set is the symmetric difference of a clopen and a first category–in fact, nowhere dense–set. See [1], e.g., for details.)

Then the Stone representation can be extended (via simple functions) to an isometry T: $L^p(\lambda, V) \cong L^p(m, V)$. (In the case $p = \infty$, where in general the simple functions are not dense, look at the dense subspace of functions taking on countably many values.)

In the scalar case each equivalence class x in $L^\infty(m)$ contains exactly one continuous function; hence $L^\infty(m) \cong CK$. Thus K is the maximal ideal space of the algebra $L^\infty[0, 1]$. The adjoint of the embedding $C[0, 1] \to L^\infty[0, 1]$, restricted to K, is a continuous, hence Borel measurable, surjection ω: $K \to [0, 1]$. Looking at the system of closed intervals with nonvoid interior, which generates the Borel σ-algebra, ω is easily seen to be inverse measure preserving and to induce the Stone representation Ψ in the sense that $\omega^{-1}(M)$ is equivalent to $\Psi(M)$ for each Borel set M.

It follows that $f \mapsto f \circ \omega$ is the isometry T mentioned above. Now let us give the example for the CK-case.

1. Example. Let W_0 be a 3-dimensional space such that there is a curve f_0: $[0, 1] \to B_0$, B_0 the unit ball of W_0, with

$$f_0(]0, 1]) \subset \operatorname{ex} B_0 \quad \text{and} \quad f_0(0) \notin \operatorname{ex} B_0.$$

(E.g., let $B_0 \subset \mathbf{R}^3$ be the convex hull of $\{(x_1, x_2, 0) | \max|x_i| \leq 1\} \cup \{(0, x_2, x_3) | x_2^2 + x_3^2 = 1\}$ and $f_0(t) := (0, \cos(\pi t/2), \sin(\pi t/2))$.) Then define

$$W := \prod_{[0,1]}^{\infty} W_0,$$

an l^∞-product of uncountably many copies of W_0, f: $[0, 1] \to W$ by

$$f(s)(t) := f_0(|s + t - 1|)$$

and x: $K \to W$ by $x := f \circ \omega$. Evidently x is continuous, and $\|x(k)\| = 1$ for all k in K. For no k in K is $x(k)$ extremal, because $x(k)(1 - \omega(k)) = f_0(0)$ is not extremal in B_0.

However, x itself is extremal. Assume $x = \frac{1}{2}(y + z)$, y and z in the unit ball of $C(K, W)$. Let $k \in K$. We have to show $x(k) = y(k)$, i.e. $x(k)(t) = y(k)(t)$ for all t in $[0, 1]$. This holds for $t \neq 1 - \omega(k)$, as in this case $x(k)(t)$ is extremal in B_0.

For $t = 1 - \omega(k)$ we choose a net (k_α) in K converging to k, with $\omega(k_\alpha) \neq \omega(k)$ for all indices α. This is possible, because $\omega^{-1}(\{\omega(k)\})$ is a closed set of measure zero, and hence has void interior. Then we have $t \neq 1 - \omega(k_\alpha)$, and so $y(k_\alpha)(t) = x(k_\alpha)(t) \to_\alpha x(k)(t)$, which in turn yields $x(k)(t) = y(k)(t)$. □

2. THEOREM. *Let K and m be as above, V a Banach space, x: $K \to V$ and $1 < p < \infty$. Then for the following conditions*
(i) *x extremal in $C(K, V)$,*
(ii) *x extremal in $L^\infty(m, V)$,*
(iii) *x extremal in $L^p(m, V)$ we have* (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

PROOF. (ii) $\Rightarrow$ (iii) is essentially contained in Theorem 1 in [6]. Assume x is extremal in the unit ball of $L^\infty(m, V)$, in particular $\|x(\cdot)\| = 1$ almost everywhere, and $x = \frac{1}{2}(y + z)$ with y and z in the unit ball of $L^p(m, V)$. Then the Clarkson inequalities [3], applied to the functions $\|y(\cdot)\|$ and $\|z(\cdot)\|$ in $L^p(m)$, yield $\|y(\cdot)\| = \|z(\cdot)\| = \|x(\cdot)\|$ almost everywhere; hence y and z are in the unit ball of $L^\infty(m, V)$.

For (i) $\Rightarrow$ (ii) assume x is extremal in the unit ball of $C(K, V)$ and $x = \frac{1}{2}(y + z)$ with y and z in the unit ball of $L^\infty(m, V)$. We look at y and z as functions rather than equivalence classes, in such a way that the equality $x(k) = \frac{1}{2}(y(k) + z(k))$ holds everywhere (without loss of generality). Now an iterative application of Egorov's theorem and of the regularity of m shows that y, as an m-almost uniform limit of continuous simple functions, is continuous on a (disjoint) union $U = \dot{\bigcup}_{n\in\mathbf{N}} C_n$, with C_n clopen and $\Sigma_{n\in\mathbf{N}}\, m(C_n) = 1$. That means y and z are continuous on the open complement U of a suitable m-null set. For each clopen subset C of U, $x|_C$ is extremal in $C(C, V)$; hence $x|_C = y|_C$. Thus $x|_U = y|_U$, which means $x = y$ in $L^\infty(m, V)$. □

3. COROLLARY. *The function f: $[0, 1] \to W$ in Example* 1, *taking no extremal vectors as values, is extremal in the unit ball of $L^p(\lambda, W)$ for* $1 < p < \infty$.

PROOF. The mapping $f \mapsto f \circ \omega$ is a linear isometry of $L^p(\lambda, W)$ onto $L^p(m, W)$. □

4. REMARKS. (a) An essential point in Theorem 2 was the fact that each Bochner measurable function is continuous on a suitable open set with a null set as complement. So it is not surprising that this theorem fails for λ instead of m: combine Johnson's result with the 4-dimensional Blumenthal-Lindenstrauss-Phelps example.

(b) The construction of the isometry preceding Example 1 applies to arbitrary finite (even infinite) measures μ,

$$T\colon L^p(\mu, V) \cong L^p(m_\mu, V),$$

and the proof of Theorem 2 holds for these measures m_μ, too. It would be interesting to know whether in general a Banach space V "recognizes" the extremal elements in $L^p(\mu, V)$ (i.e., extremal functions x satisfy condition (2) above) if and only if it recognizes those in $L^p(m_\mu, V)$. Thus, if the above isometry T is not induced by a point mapping ω as in the case $\mu = \lambda$, is it still true that the ranges of x and Tx are essentially the same?

BIBLIOGRAPHY

1. E. Behrends et al., *L^p-structure in real Banach spaces*, Lecture Notes in Math., vol. 613, Springer-Verlag, Berlin, 1977.

2. R. M. Blumenthal, J. Lindenstrauss and R. R. Phelps, *Extreme operators into $C(K)$*, Pacific J. Math. **15** (1965), 747–756.

3. J. A. Clarkson, *Uniformly convex spaces*, Trans. Amer. Math. Soc. **40** (1936), 396–414.

4. J. Diestel and J. J. Uhl, Jr., *Vector measures*, Math. Surveys, No. 15, Amer. Math. Soc., Providence, R. I., 1977.

5. J. A. Johnson, *Extreme measurable selections*, Proc. Amer. Math. Soc. **44** (1974), 107–112.

6. K. Sundaresan, *Extreme points of the unit cell in Lebesgue-Bochner function spaces*, Colloq. Math. **22** (1970), 111–119.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, SANTA BARBARA, CALIFORNIA 93106 (Current address)

MATHEMATISCHES INSTITUT, FREIE UNIVERSITAET, ARNIMALLEE 2-6, D 1000 BERLIN 33