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# KILLING VECTOR FIELDS ON COMPLETE RIEMANNIAN MANIFOLDS

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ABSTRACT. We discuss Killing vector fields with finite global norms on complete Riemannian manifolds whose Ricci curvatures are nonpositive or negative.

**1.** It is well known that if a compact Riemannian manifold has nonpositive Ricci curvature then every Killing vector field is a parallel vector field (cf. [3]). In this note, we discuss Killing vector fields with finite global norms on complete Riemannian manifolds. One of our results is that if $M$ is a complete Riemannian manifold with nonpositive Ricci curvature then every Killing vector field on $M$ with finite global norm is a parallel vector field. This is a generalization of the above well-known result. We also discuss the volume of a complete noncompact Riemannian manifold with nonpositive Ricci curvature. Our ideas are based on those of the papers of A. Andreotti and E. Vesentini [1] and, especially, H. Kitahara [2].

We shall be in the $C^\infty$-category. The manifolds considered are connected and orientable. The indices $h, i, j, k, \ldots$ run over the range $\{1, 2, \ldots, n\}$ and the Einstein summation convention will be used.

**2.** Let $M$ be an $n$-dimensional complete Riemannian manifold and $g$ (resp. $\nabla$) its Riemannian metric tensor field (resp. its Levi-Civita connection). Let $\{U: (x^1, \ldots, x^n)\}$ denote a local coordinate system on $M$. $g_{ij}$ denotes the components of $g$ and $(g^{ij})$ denotes the inverse matrix of the matrix $(g_{ij})$. We set $\nabla_i = \nabla_{\partial/\partial x^i}$ and $\nabla^i = g^{ij}\nabla_j$.

Let $\Lambda^s(M)$ be the space of all $s$-forms on $M$ and $\Lambda_0^s(M)$ the subspace of $\Lambda^s(M)$ composed of forms with compact supports. $\eta \in \Lambda^s(M)$ may be expressed locally as

$$\eta = (1/s!)\eta_{i_1 \cdots i_s}\, dx^{i_1} \wedge \cdots \wedge dx^{i_s}.$$

Let $\langle\ ,\ \rangle$ denote the local scalar product; the global scalar product $\langle\langle\ ,\ \rangle\rangle$ is defined by

$$\langle\langle \xi, \eta \rangle\rangle = \int_M \langle \xi, \eta \rangle * 1 = \int_M \xi \wedge {}^*\eta$$

for any $\xi, \eta \in \Lambda_0^s(M)$, where $*$ denotes the star operator (cf. [4]). Let $L_2^s(M)$ be the completion of $\Lambda_0^s(M)$ with respect to the scalar product $\langle\langle\ ,\ \rangle\rangle$. $d$: $\Lambda^s(M) \to \Lambda^{s+1}(M)$ denotes the exterior derivative and $\delta$: $\Lambda^s(M) \to \Lambda^{s-1}(M)$ is defined by

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$\delta = (-1)^{sn+n+1} * d *$. Then we have $\langle\langle d\xi, \eta\rangle\rangle = \langle\langle \xi, \delta\eta\rangle\rangle$ for any $\xi \in \Lambda_0^s(M)$, $\eta \in \Lambda_0^{s+1}(M)$. The Laplacian operator $\Delta$ acting on $\Lambda^*(M) = \sum_s \Lambda^s(M)$ is defined by $\Delta = d\delta + \delta d$.

For $\xi \in \Lambda^1(M)$, we have

$$(d\xi)_{ij} = \nabla_i \xi_j - \nabla_j \xi_i, \tag{2.1}$$

$$\delta\xi = -\nabla^i \xi_i, \tag{2.2}$$

$$(\Delta\xi)_i = -\nabla^j \nabla_j \xi_i - R_{\cdot i}^{h\cdot} \xi_h, \tag{2.3}$$

where $R_{\cdot i}^{h\cdot}$ denotes the components of the Ricci tensor field of $\nabla$ (cf. [4]).

Hereafter, we identify the vector fields and its dual 1-forms with respect to $g$ and they are represented by the same letters. For a vector field $\xi = \xi^i \partial/\partial x^i$, we have its dual 1-form $\xi = \xi_j dx^j = g_{ij}\xi^i dx^j$.

A vector field $\xi$ on $M$ is called a Killing vector field if $\mathfrak{L}_\xi g = 0$ where $\mathfrak{L}$ denotes the Lie derivative operator. A Killing vector field $\xi$ satisfies the following:

$$\nabla_i \xi_j + \nabla_j \xi_i = 0, \tag{2.4}$$

and, from this, we have

$$\nabla^i \xi_i = 0. \tag{2.5}$$

A Killing vector field on $M$ is called "with finite global norm" if its dual 1-form with respect to $g$ belongs in $L_2^1(M) \cap \Lambda^1(M)$.

**3.** Let $o$ be a point of $M$ and fix it. For each point $p \in M$, we denote by $\rho(p)$ the geodesic distance from $o$ to $p$. Let $B(\alpha) = \{p \in M | \rho(p) < \alpha\}$ for $\alpha > 0$. We choose a $C^\infty$-function $\mu$ on $\mathbf{R}$ (the reals) satisfying

(i) $0 \leqslant \mu(t) \leqslant 1$ on $\mathbf{R}$,

(ii) $\mu(t) = 1$ for $t \leqslant 1$,

(iii) $\mu(t) = 0$ for $t \geqslant 2$,

and we set

$$w_\alpha(p) = \mu(\rho(p)/\alpha)$$

for $\alpha = 1, 2, 3, \ldots$. Then we have

LEMMA 1 (cf. [1]). *There exists a positive number $A$, depending only on $\mu$, such that*

(i) $$\|dw_\alpha \wedge \xi\|_{B(2\alpha)}^2 \leqslant (nA/\alpha^2)\|\xi\|_{B(2\alpha)}^2,$$

(ii) $$\|dw_\alpha \wedge *\xi\|_{B(2\alpha)}^2 \leqslant (nA/\alpha^2)\|\xi\|_{B(2\alpha)}^2$$

*for any $\xi \in \Lambda^s(M)$, where*

$$\|\xi\|_{B(2\alpha)}^2 = \langle\langle \xi, \xi\rangle\rangle_{B(2\alpha)} = \int_{B(2\alpha)} \langle \xi, \xi\rangle * 1.$$

We remark that, for $\xi \in L_2^s(M) \cap \Lambda^s(M)$, $w_\alpha\xi$ belongs in $\Lambda_0^s(M)$ and $w_\alpha\xi \to \xi$ $(\alpha \to \infty)$ in the strong sense.

For any $\xi \in L_2^1(M) \cap \Lambda^1(M)$, we have

$$d\xi_\alpha = w_\alpha^2 d\xi + 2w_\alpha dw_\alpha \wedge \xi, \tag{3.1}$$

$$\delta\xi_\alpha = w_\alpha^2 \delta\xi - *(2w_\alpha dw_\alpha \wedge *\xi), \tag{3.2}$$

where $\xi_\alpha = w_\alpha^2 \xi$.

LEMMA 2 (cf. [2]). *For any* $\xi \in L_2^1(M) \cap \Lambda^1(M)$,

$$\langle\langle 2w_\alpha dw_\alpha \wedge \xi, \nabla\xi \rangle\rangle_{B(2\alpha)} + \langle\langle w_\alpha \nabla^2\xi, w_\alpha\xi \rangle\rangle_{B(2\alpha)} + \langle\langle w_\alpha \nabla\xi, w_\alpha \nabla\xi \rangle\rangle_{B(2\alpha)} = 0,$$

*where* $(\nabla^2\xi)_i = \nabla^j\nabla_j\xi_i$ *and* $(\nabla\xi)_{ij} = \nabla_i\xi_j$.

PROOF. For given $\xi$, we consider a 1-form $\eta$ defined by

$$\eta = \tfrac{1}{2} d(\langle \xi, \xi \rangle) = (\nabla_i\xi_j)\xi^j dx^i.$$

Then, $*(w_\alpha^2\eta)$ being a $(n-1)$-form with compact support in $B(2\alpha)$, we have

$$\int_M d(*(w_\alpha^2\eta)) = 0.$$

On the other hand, we have

$$d(*(w_\alpha^2\eta)) = -*\delta(w_\alpha^2\eta).$$

Thus we have

$$\int_M *\delta(w_\alpha^2\eta) = 0.$$

By (2.2) and (3.2), we have

$$\delta(w_\alpha^2\eta) = -w_\alpha^2(\nabla^i\nabla_i\xi_j)\xi^j - w_\alpha^2(\nabla_i\xi_j)(\nabla^i\xi^j) - *(2w_\alpha\, dw_\alpha \wedge *\eta).$$

Therefore we have the assertion.

Let $\xi$ be a Killing vector field on $M$ whose dual 1-form with respect to $g$ belongs in $L_2^1(M) \cap \Lambda^1(M)$. By the definition of $\Delta$, (2.2) and (2.5), we have

$$\langle\langle \Delta\xi, w_\alpha^2\xi \rangle\rangle_{B(2\alpha)} - \langle\langle \delta d\xi, w_\alpha^2\xi \rangle\rangle_{B(2\alpha)} = 0. \tag{3.3}$$

From (2.3), we have

$$\langle\langle \Delta\xi, w_\alpha^2\xi \rangle\rangle_{B(2\alpha)} = -\langle\langle w_\alpha\nabla^2\xi, w_\alpha\xi \rangle\rangle_{B(2\alpha)} + \langle\langle w_\alpha \mathfrak{R}\xi, w_\alpha\xi \rangle\rangle_{B(2\alpha)},$$

where $\mathfrak{R}$ denotes the Ricci transformation on 1-forms defined by $(\mathfrak{R}\xi)_i = -R_{\cdot i}^{h\cdot}\xi_h$.
On the other hand, by (3.1), we have

$$\langle\langle \delta d\xi, w_\alpha^2\xi \rangle\rangle_{B(2\alpha)} = \langle\langle w_\alpha d\xi, w_\alpha d\xi \rangle\rangle_{B(2\alpha)} + \langle\langle d\xi, 2w_\alpha dw_\alpha \wedge \xi \rangle\rangle_{B(2\alpha)}.$$

By (2.4),

$$\begin{aligned}
\langle d\xi, d\xi \rangle &= (1/2!)\{2(\nabla_i\xi_k)(\nabla^i\xi^k) - 2(\nabla_i\xi_k)(\nabla^k\xi^i)\} \\
&= (1/2!)\{2(\nabla_i\xi_k)(\nabla^i\xi^k) + 2(\nabla_i\xi_k)(\nabla^i\xi^k)\} \\
&= (1/2!)4(\nabla_i\xi_k)(\nabla^i\xi^k) \\
&= 4\langle \nabla\xi, \nabla\xi \rangle
\end{aligned}$$

and we have

$$\|w_\alpha d\xi\|_{B(2\alpha)}^2 = 4\|w_\alpha \nabla\xi\|_{B(2\alpha)}^2.$$

By the Schwarz inequality, Lemma 1 and the above fact, we have

$$\begin{aligned}|\langle\langle d\xi, 2w_\alpha dw_\alpha \wedge \xi\rangle\rangle_{B(2\alpha)}| &\leqslant \|w_\alpha d\xi\|_{B(2\alpha)}\|2dw_\alpha \wedge \xi\|_{B(2\alpha)} \\ &\leqslant \tfrac{1}{2}\left(\|w_\alpha d\xi\|^2_{B(2\alpha)} + 4\|dw_\alpha \wedge \xi\|^2_{B(2\alpha)}\right) \\ &\leqslant \tfrac{1}{2}\left(4\|w_\alpha \nabla\xi\|^2_{B(2\alpha)} + (4nA/\alpha^2)\|\xi\|^2_{B(2\alpha)}\right).\end{aligned}$$

Thus we have, from (3.3),

$$\begin{aligned}\langle\langle w_\alpha \mathfrak{R}\xi, w_\alpha\xi\rangle\rangle_{B(2\alpha)} &= \langle\langle w_\alpha \nabla^2\xi, w_\alpha\xi\rangle\rangle_{B(2\alpha)} + \langle\langle w_\alpha d\xi, w_\alpha d\xi\rangle\rangle_{B(2\alpha)} \\ &\quad + \langle\langle d\xi, 2w_\alpha\, dw_\alpha \wedge \xi\rangle\rangle_{B(2\alpha)} \\ &\geqslant \langle\langle w_\alpha \nabla^2\xi, w_\alpha\xi\rangle\rangle_{B(2\alpha)} + 4\|w_\alpha\nabla\xi\|^2_{B(2\alpha)} \\ &\quad - \tfrac{1}{2}\left(4\|w_\alpha\nabla\xi\|^2_{B(2\alpha)} + (4nA/\alpha^2)\|\xi\|^2_{B(2\alpha)}\right)\end{aligned}$$

(by Lemma 2)

$$\begin{aligned}&= -\langle\langle w_\alpha\nabla\xi, w_\alpha\nabla\xi\rangle\rangle_{B(2\alpha)} - \langle\langle 2w_\alpha dw_\alpha \wedge \xi, \nabla\xi\rangle\rangle_{B(2\alpha)} \\ &\quad + 4\|w_\alpha\nabla\xi\|^2_{B(2\alpha)} - \tfrac{1}{2}\left(4\|w_\alpha\nabla\xi\|^2_{B(2\alpha)} + (4nA/\alpha^2)\|\xi\|^2_{B(2\alpha)}\right)\end{aligned}$$

(by the Schwarz inequality and Lemma 1)

$$\begin{aligned}&\geqslant -\|w_\alpha\nabla\xi\|^2_{B(2\alpha)} - \tfrac{1}{2}\left(\|w_\alpha\nabla\xi\|^2_{B(2\alpha)} + (4nA/\alpha^2)\|\xi\|^2_{B(2\alpha)}\right) \\ &\quad + 4\|w_\alpha\nabla\xi\|^2_{B(2\alpha)} - \tfrac{1}{2}\left(4\|w_\alpha\nabla\xi\|^2_{B(2\alpha)} + (4nA/\alpha^2)\|\xi\|^2_{B(2\alpha)}\right).\end{aligned}$$

Therefore we have

$$\langle\langle w_\alpha \mathfrak{R}\xi, w_\alpha\xi\rangle\rangle_{B(2\alpha)} \geqslant \tfrac{1}{2}\|w_\alpha\nabla\xi\|^2_{B(2\alpha)} - (4nA/\alpha^2)\|\xi\|^2_{B(2\alpha)}.$$

Letting $\alpha \to \infty$, we have

LEMMA 3. *Let $\xi$ be a Killing vector field on $M$ with finite global norm. If $\limsup_{\alpha\to\infty} \langle\langle w_\alpha \mathfrak{R}\xi, w_\alpha\xi\rangle\rangle_{B(2\alpha)} < \infty$, then*

$$\limsup_{\alpha\to\infty} \langle\langle w_\alpha \mathfrak{R}\xi, w_\alpha\xi\rangle\rangle_{B(2\alpha)} \geqslant \tfrac{1}{2}\|\nabla\xi\|^2.$$

THEOREM 1. *If $M$ is a complete Riemannian manifold with nonpositive Ricci curvature, then every Killing vector field on $M$ with finite global norm is a parallel vector field.*

PROOF. By the nonpositivity of Ricci curvature, we have

$$\limsup_{\alpha\to\infty} \langle\langle w_\alpha \mathfrak{R}\xi, w_\alpha\xi\rangle\rangle_{B(2\alpha)} \leqslant 0$$

for any Killing vector field $\xi$ on $M$ with finite global norm. From Lemma 3, we have $\nabla\xi = 0$.

Since the length of a parallel vector field is constant, we have

COROLLARY 1. *Let $M$ be a complete noncompact Riemannian manifold with nonpositive Ricci curvature. If there exists a nontrivial Killing vector field on $M$ with finite global norm, then the volume of $M$ is finite.*

The following example illustrates the role of the hypothesis that $M$ has nonpositive Ricci curvature in the above results.

EXAMPLE 1. We take four real numbers $a_1$, $a_2$, $a_3$ and $a_4$ such that $0 < a_1 < a_2 < a_3 < a_4 < 1$ and fix them. We consider two $C^\infty$-functions $h_1, h_2$: $(0, \infty) \to \mathbf{R}$ satisfying $0 \leqslant h_i(r) \leqslant 1$ $(i = 1, 2)$ for $0 < r$ and

$$h_1(r) = 1, \quad h_2(r) = 0 \quad \text{for } 0 < r \leqslant a_2,$$
$$h_1(r) = 0, \quad h_2(r) = 1 \quad \text{for } a_3 \leqslant r.$$

We define functions $f_i$, $g_i$ $(i = 1, 2)$ as follows; $f_1(r) = h_1(r)r^{-2}(\log r)^{-2}$, $g_1(r) = h_1(r)(\log r)^{-2}$ for $0 < r < a_4$ and $f_2(r) = h_2(r)$, $g_2(r) = h_2(r)r^{-4/3}$ for $a_1 < r$. Then we set

$$\begin{array}{llll} F_1(r) = f_1(r) & (0 < r \leqslant a_3), & = 0 & (a_3 < r), \\ F_2(r) = 0 & (0 < r < a_2), & = f_2(r) & (a_2 \leqslant r), \\ G_1(r) = g_1(r) & (0 < r \leqslant a_3), & = 0 & (a_3 < r), \\ G_2(r) = 0 & (0 < r < a_2), & = g_2(r) & (a_2 \leqslant r), \end{array}$$

and

$$F(r) = F_1(r) + F_2(r), \qquad G(r) = G_1(r) + G_2(r) \quad \text{for } 0 < r.$$

The functions $F$ and $G$ are $C^\infty$ and $F(r), G(r) > 0$ for $0 < r$.

Let $M = R^2 - \{(0, 0)\} = \{(r, \theta) | 0 < r,\ 0 \leqslant \theta < 2\pi\}$ and $ds^2 = F(r)(dr)^2 + G(r)(d\theta)^2$. Then $(M, ds^2)$ is a complete Riemannian manifold. A vector field $\xi = \partial/\partial\theta$ on $M$ is a Killing vector field with respect to the Riemannian metric $ds^2$. Since $\int_0^{a_2} r^{-1}(\log r)^{-N}\, dr < \infty$ $(N = 2, 3, \ldots)$, $\int_{a_3}^\infty r^{-2/3}\, dr = \infty$ and $\int_{a_3}^\infty r^{-L}\, dr < \infty$ $(1 < L)$, we have that the volume of $M$ is infinite, $\|\xi\|$ is finite and $0 < \langle\langle \mathfrak{R}\xi, \xi\rangle\rangle < \infty$.

REMARK TO COROLLARY 1. Every complete noncompact Riemannian manifold with nonnegative Ricci curvature has infinite volume (cf. S.-T. Yau [Indiana Univ. Math. J. **25** (1976), 659–670] and E. Calabi [Notices Amer. Math. Soc. **22** (1975), A205]).

THEOREM 2. *If $M$ is a complete Riemannian manifold with negative Ricci curvature, then there is no nontrivial Killing vector field on $M$ with finite global norm.*

PROOF. Let $\xi$ be a Killing vector field on $M$ with finite global norm. By the negativity of Ricci curvature, we have $\langle\langle w_\alpha \mathfrak{R}\xi, w_\alpha\xi\rangle\rangle_{B(2\alpha)} \leqslant 0$ for every $\alpha$. From Lemma 3,

$$0 \geqslant \limsup_{\alpha\to\infty} \langle\langle w_\alpha \mathfrak{R}\xi, w_\alpha\xi\rangle\rangle_{B(2\alpha)} \geqslant \tfrac{1}{2}\|\nabla\xi\|^2 \geqslant 0.$$

Thus we have $\langle\langle w_\alpha \mathfrak{R}\xi, w_\alpha\xi\rangle\rangle_{B(2\alpha)} = 0$ for every $\alpha$. By the negativity of Ricci curvature, $\xi = 0$.

REMARK. There is a similar discussion for holomorphic vector fields on complete Kähler manifolds with finite global norms [**5**].

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