PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 84, Number 1, January 1982

THE COHESIVE PROPERTY

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ABSTRACT. We introduce the concept of cohesive families of neighborhood bases. We thereby obtain conditions necessary and sufficient to ensure that a separable space be second countable, and sufficiency conditions for complete collectionwise normality. As by-products we obtain metrizability criteria. We prove, e.g., that a T_1 space is metrizable iff it has a refined development $\{G_n: n \in N\}$ such that $\{B_p: p \in X\}$ with $B_p = \{\mathrm{St}(p, G_n): n \in N\}$ is cohesive.

1. Preliminaries. A basic result in topology states that any second countable space is separable, whereas the converse is false. The question as to precisely what property must be added to separability to produce second countability appears to remain unanswered. We provide an answer via the concept of "cohesive" families, a concept we define shortly. We then further apply this cohesive property to obtain criteria for collectionwise normality and metrizability, and a characterization of metrizability in terms of uniformly continuous semimetrics. First, however, we comment on terminology and notation.

If p is a point in a topological space (X, T), a neighborhood of p is a set $V(p)$ such that $p \in 0 \subset V(p) \subset X$ for some $0 \in T$. We refer to a neighborhood basis B_p at p as a "local base at p" if $B_p \subset T$. And B_p is "monotone" if B_p is linearly ordered by set inclusion, with the usual requirement that $V_{n+1}(p) \subset V_n(p)$ for $n \in N$ when B_p is countable (N denotes the set of positive integers). A T_1 space (X, T) is a semimetric space iff there is a function d on $X \times X$ into the nonnegative reals (called a semimetric) such that $d(x, y) = d(y, x)$, $d(x, y) = 0$ iff $x = y$, and d is compatible with the topology T (i.e., if $M \subset X$, $x \in \overline{M}$ iff $\inf\{d(x, y): y \in M\} = 0$). We let $S(x, r)$ denote the set $\{y \in X: d(x, y) < r\}$. And (X, T) is developable iff there is a sequence $G = \{G_n: n \in N\}$ of open covers of X such that $\{\mathrm{St}(x, G_n): n \in N\}$ is a local base at x for each $x \in X$. G is called a development for X.

2. The cohesive concept and applications. Let (X, T) be a space and let B be any basis for T. If $B_p = \{0 \in B: p \in 0\}$, then the family $\{B_p: p \in X\}$ is a "cohesive" family.

DEFINITION 2.1. Let (X, T) be a topological space, and for each $p \in X$ let B_p be a neighborhood basis at p. The family $\{B_x: x \in X\}$ is cohesive at $p \in X$ iff the following obtains.

$(*)$ If $0 \in T$ and $p \in 0$, there exists $V \in B_p$ such that $x \in V$ implies that $p \in U$ and $U \subset 0$ for at least one $U \in B_x$.

Received by the editors August 29, 1980.
1980 *Mathematics Subject Classification*. Primary 54D15, 54E25, 54E30, 54E35.
Key words and phrases. Cohesive families, semimetric, developable, metrizable.

We say that the family is cohesive iff it is cohesive at each point of X.

As noted above, any space has cohesive families of neighborhood bases. We can sometimes be selective.

EXAMPLE 2.2. If (X, d) is a metric space and $B_p = \{S(p, 1/n): n \in N\}$, then $\{B_p: p \in X\}$ is a cohesive family in which each B_p is monotone and countable.

Of course, very nice spaces may have monotone countable local bases which generate noncohesive families.

EXAMPLE 2.3. Let $X = R$, the reals with the usual topology, and Z the set of integers. If $x \in (i, i + 1)$ for some $i \in Z$, let $B_x = \{(x - 1/n, x + 1/n) \cap (i, i + 1): n \in N\}$, and for $x \in Z$ let $B_x = \{(x - 1/n, x + 1/n): n \in N\}$. Then each B_x is a local base and the family $\{B_x: x \in X\}$ is cohesive on $X - Z$ but at no point of Z.

The next example motivates our first result.

EXAMPLE 2.4. If $X = R$ and T is the lower limit topology generated by the half-open intervals $[a, b)$, then (X, T) is first countable and separable (and paracompact, [6, p. 175]), but not second countable. The following theorem tells us that if B_p is a countable local base for each p, then the family $\{B_p: p \in X\}$ cannot be cohesive.

THEOREM 2.5. *A topological space (X, T) is second countable if and only if it is separable and has a countable local base B_p at each $p \in X$ such that the family $\{B_p: p \in X\}$ is cohesive.*

PROOF. (*Necessity.*) Since any second countable space is separable, we need only prove the cohesive property. So let B be a countable base for T. For $p \in X$ let $B_p = \{V: V \in B \text{ and } p \in V\}$. Clearly, B_p is a countable local base. To see that $\{B_p: p \in X\}$ is cohesive, let $p \in 0 \in T$. Since B is a basis for T, there exists $V \in B$ such that $p \in V$ and $V \subset 0$. Now $V \in B_p$ by definition of B_p. But since $V \in B_x$ for any $x \in V$, condition $(*)$ in the definition of cohesive families is trivially satisfied with $U = V$.

(*Sufficiency.*) Let $\{B_p: p \in X\}$ be a cohesive family assured by the hypothesis, and let D be a countable dense subset of X. Define $B = \bigcup \{B_p: p \in D\}$. Clearly, B is countable. We assert that B is a basis for T. First note that $B \subset T$. Now let $0 \in T$ and $p \in 0$. Let $V \in B_p$ which satisfies property $(*)$ with respect to the given 0 and p. Since D is dense and V is open, there exists $x \in D \cap V$. By $(*)$ $p \in U \subset 0$ for some $U \in B_x$. But $U \in B_x$ and $x \in D$ imply that $U \in B$, and the theorem is proved.

We next consider the cohesive concept in the context of monotone neighborhood bases B_p, with the reminder that a T_1 space is collectionwise normal iff every discrete collection of sets can be covered by a pairwise disjoint collection of open sets, each of which covers just one of the original sets.

THEOREM 2.6. *Let (X, T) be a T_1 space which has a neighborhood basis B_p at each $p \in X$ such that $B = \{B_p: p \in X\}$ is cohesive. If X has a dense subset D such that B_p is monotone for each $p \in D$, then (X, T) is collectionwise normal.*

PROOF. Let F be a discrete collection of subsets of X. For each $x \in \bigcup F$ we let C_x denote the unique set in F which contains x. Thus $C_x \cap C_y \neq \emptyset$ iff $C_x = C_y$. Since F is discrete, for each $x \in \bigcup F$ there exists $V_x \in B_x$ such that

$$V_x \cap (\bigcup F) \subset C_x, \tag{1}$$

and since B is cohesive, for each $x \in \bigcup F$ we can choose $U_x \in B_x$ such that (*) $z \in U_x$ implies that $x \in V_z^x \subset V_x$ for some $V_z^x \in B_z$.

We now let $O_x = \bigcup \{\mathrm{Int}(U_p): p \in C_x\}$ for all $x \in \bigcup F$. Since each p determines a unique set U_p,

$$O_x = O_y \quad \text{if } C_x = C_y. \tag{2}$$

Clearly $C_x \subset O_x$ and each O_x is open. Moreover, $O_x \cap (\bigcup F) = C_x$ by (1), since $U_p \subset V_p$. So to prove that X is collectionwise normal, we have yet to show that the members of $\{O_x: x \in \bigcup F\}$ are pairwise disjoint.

To this end, we suppose that $O_x \cap O_y \neq \emptyset$ for some x, y and we show that $O_x = O_y$. The definition of O_x and O_y yields $p \in C_x$ and $q \in C_y$ such that $\mathrm{Int}(U_p) \subset O_x$, $\mathrm{Int}(U_q) \subset O_y$, and $\mathrm{Int}(U_p) \cap \mathrm{Int}(U_q) \neq \emptyset$. Thus there exists $z \in U_p \cap U_q \cap D$, since D is dense in X. Then by (*), $z \in U_p$ implies there exists $V_z^p \in B_z$ such that $p \in V_z^p \subset V_p$, and $z \in U_q$ implies there exists $V_z^q \in B_z$ such that $q \in V_z^q \subset V_q$.

But $z \in D$ so that B_z is monotone, and we may assume w.l.o.g that $V_z^q \subset V_z^p$. By the above we therefore have $p, q \in V_p$. Since $p \in C_x$ and $q \in C_y$, $V_p \cap C_x \neq \emptyset$ and $V_p \cap C_y \neq \emptyset$. But $V_p \cap (\bigcup F) \subset C_p$ (see (1)), so $C_p \cap C_x \neq \emptyset$ and $C_p \cap C_y \neq \emptyset$. Thus $C_x = C_p = C_y$, and (2) implies $O_x = O_y$. $\square$

Since the properties of a neighborhood basis B_p being monotone and the family $\{B_p: p \in X\}$ being cohesive are clearly hereditary, we can say

COROLLARY 2.7. *A T_1 space X which has a monotone neighborhood basis B_p at each point p such that $\{B_p: p \in X\}$ is cohesive is hereditarily collectionwise normal (or equivalently, completely collectionwise normal).*

Corollary 2.7 and a result by McAuley [**4**] imply

COROLLARY 2.8. *Any semimetric space (X, d) in which the family $\{B_p: p \in X\}$ with $B_p = \{S(p, 1/n): n \in N\}$ is cohesive is completely collectionwise normal and paracompact.*

On the other hand, a well-known example of McAuley (see [**5** or **6**, p. 175]) demonstrates that a semimetric space may be paracompact and even hereditarily separable although the neighborhoods $S(p, 1/n)$ do not generate a cohesive family. That the $1/n$ neighborhoods do not generate a cohesive family follows from Theorem 2.5 and the fact that McAuley's space is separable but not second countable. Now McAuley's space is not developable. Another well-known example, the "tangent disc" toplogy [**6**, p. 176] is a regular developable space which is not normal and therefore, by Theorem 2.6, the family $\{B_p: p \in X\}$ with $B_p = \{\mathrm{St}(p, G_n): n \in N\}$ is not cohesive ($\mathrm{St}(p, G_n) = \bigcup \{0 \in G_n: p \in 0\}$). However, if we combine the cohesive property with developability, we obtain metrizability.

COROLLARY 2.9. *A T_1 space X is metrizable iff there is a refined development* $\{G_n: n \in N\}$ *of* X *such that* $\{B_p: p \in X\}$ *with* $B_p = \{\mathrm{St}(p, G_n): n \in N\}$ *is cohesive.*

PROOF. The "necessity" is immediate. To verify " sufficiency" note that since G_{n+1} refines G_n, each B_p is monotone, so that X is collectionwise normal. But as is well known, any collectionwise normal developable space is metrizable.

3. Metrizability criteria. A theorem in [3] states

THEOREM [3]. *A T_3 space (X, T) is metrizable iff there is a semimetric d compatible with T such that*

(i) $\lim_n d(x_n, y_n) = \lim_n d(x_n, p) = 0$ *implies* $\lim_n d(y_n, p) = 0$.

The T_3 requirement in the above result is redundant. For suppose that (X, T) is a T_1 space and d is a semimetric compatible with T for which (i) above holds. Thus $S(p, 1/n)$ is indeed a neighborhood of p, and it is a simple matter to show that for any $p \in X$ and $n \in N$, there exists $k = K(p, n) \geqslant n$ such that $p \in S(x, 1/k) \subset S(p, 1/n)$ for all $x \in S(p, 1/k)$ (remember, $x \in S(p, 1/k)$ iff $p \in S(x, 1/k)$ since d is "symmetric"). Thus if $B_p = \{S(p, 1/n): n \in N\}$, then $\{B_p: p \in X\}$ is cohesive—in a uniform way. Consequently, Theorem 2.6 applies, so that (X, T) is certainly T_3 and we have

PROPOSITION 3.1. *A T_1 space (X, T) is metrizable iff there is a semimetric d compatible with T such that*

(i) $\lim_n d(x_n, p) = \lim_n d(x_n, y_n) = 0$ *implies* $\lim d(y_n, p) = 0$.

Now a T_1 space (X, T) which has a continuous semimetric compatible with T is developable but need not even be normal [2]. The next result, which we believe (surprisingly) to be new, tells us what to add to continuity to obtain metrizability.

THEOREM 3.2. *A T_1 space (X, T) is metrizable iff there is a semimetric d compatible with T which is uniformly continuous.*

A semimetric d is continuous at (a, b) iff for any $\varepsilon > 0$ there is a $\delta > 0$ such that (ii) $|d(x, y) - d(a, b)| < \varepsilon$ *if* $\max\{d(x, a), d(y, b)\} < \delta$. *Of course, d is uniformly continuous if* $\exists \delta > 0$ *such that* (ii) *holds for any* (a, b) *and* (x, y) *in* $X \times X$.

PROOF (OF THEOREM 3.2). If (X, T) is metrizable and d is a metric compatible with T, the triangle inequality yields

$$|d(x,y) - d(a, b)| \leqslant d(x, a) + d(y, b) \leqslant 2\max\{d(x, a), d(y, b)\}$$

so that (ii) above is satisfied for any (x, y) and (a, b) and any ε if $\delta = \varepsilon/2$; thus d is uniformly continuous.

Conversely, suppose that d is uniformly continuous and that $d(x_n, p) \to 0$ and $d(x_n, y_n) \to 0$, and let $\varepsilon > 0$. By uniform continuity $\exists \delta > 0$ such that $d(p, y_n) = |d(y_n, p) - d(x_n, x_n)| < \varepsilon$ if (iii) $\max\{d(y_n, x_n), d(p, x_n)\} < \delta$. Since $d(x_n, y_n) \to 0$ and $d(x_n, p) \to 0$, there exists k such that (iii) holds for $n \geqslant k$; i.e., $d(y_n, p) < \varepsilon$ if $n \geqslant k$. Thus $\lim d(y_n, p) = 0$, and (X, T) is metrizable by Proposition 3.1. $\square$

(For a study of continuity and uniform continuity in the context of generalized metric spaces see **[1]**.)

In the comments preceding Proposition 3.1 we observed that the requirement (i) induced a local uniform cohesiveness on the $1/n$ neighborhoods, a concept we now utilize.

THEOREM 3.3. *A T_1 space (X, T) is metrizable iff there is a countable monotone local base $\{V_n(p): n \in N\}$ at each $p \in X$ such that* (**) *for each $n \in N$, $\exists k = k(n, p) \geqslant n$ such that $p \in V_k(x) \subset V_n(p)$ for all $x \in V_k(p)$.*

PROOF. The "necessity" follows immediately with $V_n(p) = S(p, 1/n)$. To prove that the condition is sufficient, for each $n \in N$ we let $G_n = \{V_{k(n,x)}(x): x \in X\}$ where $k = k(n, x)$ is chosen to satisfy (**). We assert that $G = \{G_n: n \in N\}$ is a development. Clearly, each G_n is an open cover of X, so we have yet to show that $\{\mathrm{St}(p, G_n): n \in N\}$ is a local base at p for $p \in X$. To see this, let $p \in 0$ for some $0 \in T$. Since B_p is a local base at p, we can choose n such that $V_n(p) \subset 0$. We prove that $\mathrm{St}(p, G_{k(n,p)}) \subset V_n(p)$. Note that

$$\mathrm{St}(p, G_{k(n,p)}) = \bigcup \{V_{k(k(n,p),x)}(x): p \in V_{k(k(n,p),x)}(x)\}. \tag{1}$$

If $p \in V_{k(k(n,p),x)}(x)$, the definition of $k(k(n, p), x)$ and (**) imply $x \in V_{k(k(n,p),x)}(p) \subset V_{k(n,p)}(x)$. But

$$k(k(n, p), x) \geqslant k(n, p). \tag{2}$$

Thus by monotonicity $x \in V_{k(n,p)}(p)$, so the designation of $k(n, p)$ and (**) imply $p \in V_{k(n,p)}(x) \subset V_n(p)$. Thus (2) yields $V_{k(k(n,p),x)}(x) \subset V_n(p)$, and by (1) we have $\mathrm{St}(p, G_{k(n,p)}) \subset V_n(p)$ as desired; i.e., (X, T) is developable. But by Theorem 2.6 (X, T) is also collectionwise normal and therefore metrizable, since any collectionwise normal developable space is metrizable. □

In closing we note that since Urysohn's Theorem tells us that any regular second countable T_1 space is metrizable, we can apply Theorems 2.5 and 2.6 to conclude

COROLLARY 3.4. *A separable T_1 space X is metrizable iff there is a countable monotone local base B_p at each $p \in X$ such that $\{B_p: p \in X\}$ is cohesive.*

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