PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 84, Number 2, February 1982

A SIMPLE PROOF OF THE EXTENSION THEOREM OF SEQUENCES OF DIVIDED POWERS IN CHARACTERISTIC p

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ABSTRACT. Using the idea of relative Hopf modules, a short proof of the extension theorem of sequences of divided powers in irreducible cocommutative Hopf algebras over a field of characteristic $p > 0$ is presented.

Let k be a field of characteristic $p > 0$. Let H be an irreducible cocommutative Hopf algebra over κ. The V-map for H [1, (4.1)]

$$V\colon H \to k^{1/p} \otimes H$$

is a $\frac{1}{p}$-linear Hopf algebra map [1, p. 279] with kernel HL where $L = P(H)$, the primitives in H [4, Theorem 1]. We define a descending set of Hopf subalgebras $\{V^n(H)\}_{n \geq 0}$ inductively as follows: $V^0(H) = H$, $V^n(H) = V(V^{n-1}(H)) \cap H$. ($V^1(H) = V(H) \cap H$ is different from $V(H)$.) Since $V(H)$ is a $k^{1/p}$-Hopf subalgebra of $k^{1/p} \otimes H$, it is easy to check that each $V^n(H)$ is a k-Hopf subalgebra of H. An element $x \in H$ has coheight n if $x \in V^n(H)$. For each integer $e > 0$, the integer $\|e\| \geq 0$ is defined by

$$p^{\|e\|} \leq e < p^{\|e\|+1}.$$

A set of elements $x_0 = 1, x_1, \ldots, x_n$ (n finite) in H is called an *n-sequence of divided powers* if

$$\Delta(x_i) = \sum_{j=0}^{i} x_j \otimes x_{i-j}, \qquad 0 \leq i \leq n.$$

THEOREM A [4, LEMMA 7; AND 2, THEOREM 2]. *Let $t < p^{n+1}$ and let $x_0, x_1, \ldots, x_{t-1}$ be a sequence of divided powers in H where x_i has coheight $n - \|i\|$, $0 \leq i < t$. There is an element x_t in H of coheight $n - \|t\|$ such that $x_0, x_1, \ldots, x_{t-1}, x_t$ is a sequence of divided powers.*

The following extension theorem of sequences of divided powers is a key lemma to determine the coalgebra structure of H [4, Theorems 2 and 3].

The original proof of Sweedler, which consists of several steps, is done by induction on n and t. In the following, we give an alternative proof, where we do not use induction, but the idea of relative Hopf modules [5] instead.

Received by the editors November 18, 1980.
1980 *Mathematics Subject Classification*. Primary 16A24.
Key words and phrases. Hopf algebra, sequence of divided powers.
[1] Supported in part by NSF Grant MCS-77-18723 A03.

PROOF. Replacing H by $V^{n-\|t\|}(H)$, we may assume $n = \|t\|$. Let $\tilde{H}$ be the free k-algebra generated by H and one indeterminate z. Thus, if A is a k-algebra and φ: $H \to A$ an algebra map, then for any $a \in A$, there is a unique algebra map $\tilde{\varphi}$: $\tilde{H} \to A$ such that $\tilde{\varphi}|H = \varphi$ and $\tilde{\varphi}(z) = a$. Using this universal mapping property, define algebra maps

$$\tilde{\Delta}: \tilde{H} \to \tilde{H} \otimes \tilde{H}, \qquad \tilde{\varepsilon}: \tilde{H} \to k$$

by the rule: $\tilde{\Delta}|H = \Delta$ (comultiplication of H), $\tilde{\Delta}(z) = z \otimes 1 + 1 \otimes z + \sum_{i=1}^{t-1} x_i \otimes x_{t-i}$, $\tilde{\varepsilon}|H = \varepsilon$ (augmentation of H), $\tilde{\varepsilon}(z) = 0$. Then $(\tilde{H}, \tilde{\Delta}, \tilde{\varepsilon})$ is an irreducible cocommutative Hopf algebra containing H as a Hopf subalgebra. Since x_0, $x_1, \ldots, x_{t-1}$, z is a t-sequence of divided powers in $\tilde{H}$, $V(z) = 0$ if $p \nmid t$ and $V(z) = x_s$ if $t = ps$. In the latter case, x_s has coheight $\|t\| - \|s\| = 1$. Hence $V(z) \in V(H)$. Since V is a semilinear Hopf algebra map, it follows that $V(\tilde{H}) = V(H)$. Let $\tilde{L} = P(\tilde{H})$, the primitives in $\tilde{H}$. Let U (resp. $\tilde{U}$) be the restricted universal enveloping algebra of L (resp. $\tilde{L}$). Then U (resp. $\tilde{U}$) is a Hopf subalgebra of H (resp. $\tilde{H}$) [3, Proposition 13.2.3]. We claim that the multiplication in $\tilde{H}$ induces an isomorphism

$$H \otimes_U \tilde{U} \xrightarrow{\sim} \tilde{H}.$$

Indeed, both sides are right $(\tilde{H}, \tilde{U})$-Hopf modules [5, p. 454] and the map is a homomorphism. Since $\tilde{H}$ is irreducible, $\tilde{H}$ is a free left (or right) $\tilde{U}$-module [5, Proposition 3]. Hence the category of right $(\tilde{H}, \tilde{U})$-Hopf modules is equivalent to the category of right $\tilde{H}/\tilde{H}\tilde{L}$-comodules [5, Theorem 1], where the equivalence is given by $M \mapsto M/M\tilde{L}$. If we apply this equivalence functor to the above homomorphism, we get the canonical map $H/HL \to \tilde{H}/\tilde{H}\tilde{L}$ which is an isomorphism, since $H/HL \simeq V(H)$, $\tilde{H}/\tilde{H}\tilde{L} \simeq V(\tilde{H})$ and $V(\tilde{H}) = V(H)$. This proves our claim.

Let X be a basis of $\tilde{L}$ modulo L. Let Λ be the set of all functions from X to $\{0, 1, \ldots, p-1\}$ with finite support. Give a total order on X. For each f in Λ, put

$$[f] = \frac{c_1^{e_1} \cdots c_n^{e_n}}{e_1! \cdots e_n!} \quad \text{and} \quad |f| = e_1 + \cdots + e_n$$

where $\{c_1, \ldots, c_n\}$ is the support of f with $c_1 < \cdots < c_n$ and $e_i = f(c_i)$. Then $\{[f] \| f \in \Lambda\}$ is a free basis of the left U-module $\tilde{U}$ (Poincaré-Birkhoff-Witt), hence of the left H-module $\tilde{H}$, and we have

$$\Delta[f] = \sum_{f=g+h} [g] \otimes [h].$$

Write $z = \sum_{f \in \Lambda} z_f[f]$, $z_f \in H$. Then

$$\tilde{\Delta}(z) = \sum \Delta(z_{g+h})([g] \otimes [h])$$

where the sum is taken over the set of all $g, h \in \Lambda$ with $g + h \in \Lambda$. Since $\tilde{\Delta}(z) = z \otimes 1 + 1 \otimes z + \sum_{i=1}^{t-1} x_i \otimes x_{t-i}$, and $\{[g] \otimes [h] \| g, h \in \Lambda\}$ is a free basis of the left $H \otimes H$-module $\tilde{H} \otimes \tilde{H}$, it follows from comparison of the coefficients that $z_f = 0$ for $|f| > 1$ and $z_f \in k$ for $|f| = 1$. Put $x_t = z - \sum_{|f|=1} z_f[f]$. Then $x_t \in H$ and $\tilde{\Delta}(z) - z \otimes 1 - 1 \otimes z = \Delta(x_t) - x_t \otimes 1 - 1 \otimes x_t$. Hence x_0, $x_1, \ldots, x_{t-1}, x_t$ is a sequence of divided powers in H. Q.E.D.

The above idea of proof yields more general results. Note that we merely used the fact that $V(H) = V(\tilde{H})$ in the latter part of the above proof. Hence, what we proved actually is the following

THEOREM B. *Let $\tilde{H}$ be an irreducible cocommutative Hopf algebra and let $H \subset \tilde{H}$ be a Hopf subalgebra. Assume $V(H) = V(\tilde{H})$. If $z \in \tilde{H}$ satisfies*

$$\Delta(z) - z \otimes 1 - 1 \otimes z \in H \otimes H$$

there is an element $x \in H$ such that

$$\Delta(z) - z \otimes 1 - 1 \otimes z = \Delta(x) - x \otimes 1 - 1 \otimes x.$$

It is enough to assume $V(z) \in V(H)$ instead of $V(H) = V(\tilde{H})$. (Replace $\tilde{H}$ by the Hopf subalgebra generated by H and z.)

The above theorem can be interpreted as a cohomological vanishing theorem of the underlying coalgebras of irreducible cocommutative Hopf algebras. To clarify the meaning, for a pointed irreducible cocommutative coalgebra C, let $C^+ = \operatorname{Ker}(\varepsilon)$ and

$$\delta: C^+ \to C^+ \otimes C^+, \qquad \delta(x) = \Delta(x) - x \otimes 1 - 1 \otimes x$$

where 1 denotes the unique group-like element of C. We want to determine the image $\delta(C^+)$. Let $\delta_n: C^+ \to \bigotimes^{n+1} C^+$ be the n times iterated δ-map. Let

$$u = \sum_i x_i \otimes y_i \in C^+ \otimes C^+$$

be an element satisfying

(a) $\sum_i x_i \otimes y_i = \sum_i y_i \otimes x_i$,

(b) $\sum_i \delta(x_i) \otimes y_i = \sum_i x_i \otimes \delta(y_i)$.

There is a pointed irreducible cocommutative coalgebra $C^u = C \oplus kz$ which contains C as a subcoalgebra and satisfies

$$\Delta(z) = z \otimes 1 + 1 \otimes z + u, \qquad \varepsilon(z) = 0.$$

Then $V(z)$ is determined by u as follows: $\sum_i \delta_{p-2}(x_i) \otimes y_i$ is a symmetric tensor in $\bigotimes^p C^+$. Let

$$v: (\text{the symmetric tensors in } \otimes^p C^+) \to k^{1/p} \otimes C^+$$

be the $\frac{1}{p}$-linear map defined [**1**, Theorem 4.1.1(a), p. 273] (where denoted by V). Put $v(u) = v(\sum_i \delta_{p-2}(x_i) \otimes y_i)$. Then $V(z)$ is precisely $v(u)$.

If C underlies a Hopf algebra, then the image $\delta(C^+)$ can be characterized as follows.

THEOREM C. *Let H be an irreducible cocommutative Hopf algebra. The image $\delta(H^+)$ is precisely the set of elements u in $H^+ \otimes H^+$ satisfying* (a), (b) *and* (c) $v(u) \in V(H)$.

PROOF. If $u = \delta(x)$ with $x \in H^+$, then u satisfies (a), (b) and $v(u) = V(x) \in V(H)$. Conversely, if u satisfies (a), (b), (c), let $\tilde{H}$ be the Hopf algebra generated by H and one indeterminate z with $\delta(z) = u$, $\varepsilon(z) = 0$. It follows from $V(z) = v(u) \in V(H)$ that $V(H) = V(\tilde{H})$. Hence $\delta(z) = \delta(x)$ for some $x \in H$ by Theorem B. Q.E.D.

Theorem A follows from Theorem C applied to $u = \sum_{i=1}^{t-1} x_i \otimes x_{t-i}$ and $V^{n-\|t\|}(H)$ as H.

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