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A TAUBERIAN THEOREM FOR STRONG ABEL SUMMABILITY TYPE

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ABSTRACT. In the present paper the author has defined a new method of strong Abel summability type $\{A, \lambda\}_m$ and obtained a necessary and sufficient type of Tauberian conditions for Σa_n to be summable $[R, \lambda, k]_m$, whenever it is summable $\{A, \lambda\}_m$. This result is analogous to one result of Flett [4].

1. Let $\{\lambda_n\}$ be an increasing sequence of nonnegative numbers tending to ∞ with n. The series Σa_n is said to be strongly summable $[R, \lambda, k]_m$, $k > 1 - \frac{1}{m}$, to the sum s [5, 8], if

$$\int_{\lambda_0}^{x} |c_\lambda^{k-1}(t) - s|^m \, dt = o(x), \quad \text{as } x \to \infty, \tag{1.1}$$

where $c_\lambda^k(x) = \Sigma_{\lambda_v < x}(1 - \lambda_v/x)^k a_v$.

It is noteworthy to remark that in [5], Glatfeld uses a different notation.

We also write

$$A_\lambda^k(x) = \sum_{\lambda_n < x} (x - \lambda_n)^k a_n, \qquad B_\lambda^k(x) = \sum_{\lambda_n < x} (x - \lambda_n)^k \lambda_n a_n.$$

It is natural to define the series Σa_n as summable $\{A, \lambda\}_m$ $(m \geqslant 1)$ if the series $\phi(x) = \Sigma_{n=0}^\infty a_n \exp(-\lambda_n x)$ converges for all $x > 0$ and

$$\int_R^\infty \frac{|\phi(x) - s|^m}{(1 - e^{-x})^2} e^{-x} \, dx = o\left(\frac{1}{1 - e^{-R}}\right), \quad \text{as } R \to 0. \tag{1.2}$$

For in the special case $\lambda_n = n$ this reduces after an obvious change of variable, to the definition of $\{A\}_m$ given by Flett in [4]. However we can put this definition in a simpler form. In fact, (1.2) is equivalent to the assertions that as $R \to 0+$,

$$\int_R^\infty \frac{|\phi(x) - s|^m}{x^2} \, dx = o\left(\frac{1}{R}\right). \tag{1.3}$$

To prove this, we first note that, as $R \to 0+$, $1/(1 - e^{-R}) \sim \frac{1}{R}$ so that $o(1/(1 - e^{-R}))$ on the right of (1.2) can be replaced by $o(\frac{1}{R})$. Next, the assumption that $\Sigma_{n=0}^\infty a_n \exp(-\lambda_n x)$ converges for $x > 0$ implies that the sum is bounded for $x \geqslant 1$ (see [6]). Hence the integrals

$$\int_1^\infty \frac{|\phi(x) - s|^m}{(1 - e^{-x})^2} e^{-x} \, dx, \qquad \int_1^\infty \frac{|\phi(x) - s|^m}{x^2} \, dx$$

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converge. Thus, since these integrals do exist, they are constants and hence certainly $o(\frac{1}{R})$, so that it makes no difference to the validity of either (1.2) or (1.3) if we replace the integrals from R to ∞ by integrals from R to 1. But, for $0 < x \leqslant 1$,

$$\alpha/x^2 \leqslant e^{-x}/(1 - e^{-x})^2 \leqslant \beta/x^2,$$

where α, β are strictly positive absolute constants. Hence, for $0 < R < 1$,

$$\alpha\int_R^1 \frac{|\phi(x) - s|^m}{x^2}\,dx \leqslant \int_R^1 \frac{|\phi(x) - s|^m}{(1 - e^{-x})^2} e^{-x}\,dx \leqslant \beta\int_R^1 \frac{|\phi(x) - s|^m}{x^2}\,dx, \tag{1.4}$$

so that if either one of the integrals occurring in (1.4) is $o(\frac{1}{R})$ then so is the other proving the result.

REMARK. The equivalent definition (1.3) and the above proof is due to B. Kuttner.

Hyslop and Boyd [1] have shown that for $\lambda_n = n$, $[R, \lambda, k]_m$ is equivalent to $[c, k]_m$.

Throughout we denote by M a positive constant, which may be different at each occurrence.

2. Introduction. Flett [3] has established that $\sum_{n=1}^{r}|\tau_n^k|^m = o(r)$ is a Tauberian condition for $\sum a_n \in \{A\}_m$ to imply $\sum a_n \in [c, k]_m$, $m \geqslant 1$ and $k > 0$. The object of the present paper is to obtain an analogue of Flett's Theorem [3] and get a necessary and sufficient type of Tauberian condition for $\sum a_n \in \{A, \lambda\}_m$ to imply $\sum a_n \in [R, \lambda, k]_m$.

3. We establish the following.

THEOREM. *If $m \geqslant 1$ and $k > 1 - \frac{1}{m}$ and* (i) *$\sum_{n=0}^{\infty} a_n$ is summable $\{A, \lambda\}_m$, then a necessary and sufficient condition for $\sum_{n=0}^{\infty} a_n$ to be summable $[R, \lambda, k]_m$ is*

$$\int_0^X \left|\frac{B_\lambda^{k-1}(x)}{x^k}\right|^m dx = o(X), \quad \text{as } X \to \infty. \tag{3.1}$$

4. We require the following lemmas.

LEMMA 1 [2]. *If $\phi(x) = \sum_{n=0}^{\infty} a_n \exp(-\lambda_n x)$ converges for $x > 0$, then for $k \geqslant 0$, we have*

$$\phi(x) = \frac{x^{k+1}}{\Gamma(k + 1)} \int_0^\infty A_\lambda^k(t) e^{-xt}\,dt.$$

LEMMA 2. *For $\sum a_n$ to be summable $[R, \lambda, k]_m$ it is necessary and sufficient that* (i) *$\sum a_n$ is summable $[R, \lambda, k + 1]_m$ and* (ii) (3.1) *holds.*

PROOF. Let $m \geqslant 1$. Suppose first that (i) and (ii) are satisfied. We have [7]

$$x^{-k+1}A_\lambda^{k-1}(x) = x^{-k}A_\lambda^k(x) + x^{-k}B_\lambda^{k-1}(x). \tag{4.1}$$

Hence

$$\int_0^X |c_\lambda^{k-1}(x)|^m\,dx \leqslant M\int_0^X |c_\lambda^k(x)|^m\,dx + M\int_0^X \left|\frac{B_\lambda^{k-1}(x)}{x^k}\right|^m dx$$

Since $[R, \lambda, k]_m \Rightarrow [R, \lambda, k+1]_m$ [7], (i) is necessary. Again by (4.1), (3.1) follows. Hence (3.1) is necessary.

LEMMA 3. *If* $m \geqslant 1, f(x) = f(x, t)$ *and* $g(x) = g(x, t)$ *then*

$$\left(\int dt \int |f(x) + s(x)|^m \, dx\right)^{1/m} \leqslant \left(\int dt \int |f(x)|^m \, dx\right)^{1/m} + \left(\int dt \int |g(x)|^m \, dx\right)^{1/m}.$$

The result follows by double applications of Minkowski's inequality.

LEMMA 4. *If* $k > -1$, *then*

$$\int_u^\infty v^k e^{-tv} \, dv = O\left(\frac{u^k e^{-tu}}{t}\right) + O\left(\frac{e^{-tu}}{t^{k+1}}\right),$$

uniformly for all $t > 0$ *and* $u \geqslant 0$.

PROOF. If k is any real constant, then uniformly in $y \geqslant 1$,

$$\int_y^\infty w^k e^{-w} \, dw = O(y^k e^{-y}). \tag{4.2}$$

This may be proved by observing that for sufficiently large y, say $y > Y$,

$$y^k e^{-y} = -\int_y^\infty \frac{d}{dw}(w^k e^{-w}) \, dw > c \int_y^\infty w^k e^{-w} \, dw,$$

where c is a strictly positive constant. But for fixed Y, (4.2) certainly holds in the range $1 \leqslant y \leqslant Y$, since, in that range, $\int_y^\infty w^k e^{-w} \, dw$ and $y^k e^{-y}$ each lies between two (strictly) positive constants. By an obvious change of variables, we deduce that, uniformly in $tu \geqslant 1$,

$$\int_u^\infty v^k e^{-tv} \, dv = O\left(\frac{u^k e^{-tu}}{t}\right). \tag{4.3}$$

Let us now suppose $k > -1$. Then

$$\int_u^\infty v^k e^{-tv} \, dv \leqslant \int_0^\infty v^k e^{-tv} \, dv = \frac{\Gamma(k+1)}{t^{k+1}}.$$

If $tu < 1$ then $1 \leqslant e \cdot e^{-tu}$ and hence, uniformly in $tu < 1$,

$$\int_u^\infty v^k e^{-tv} \, dv = O\left(\frac{e^{-tu}}{t^{k+1}}\right). \tag{4.4}$$

Since (4.3) holds for $tu \geqslant 1$ and (4.4) for $tu < 1$ we have uniformly for all $t > 0$, $u > 0$,

$$\int_u^\infty v^k e^{-tv} \, dv = O\left(\frac{u^k e^{-tu}}{t}\right) + O\left(\frac{e^{-tu}}{t^{k+1}}\right).$$

LEMMA 5. *If* $0 < m\mu < 1$, *then* (3.1) *implies*

$$\int_0^X x^{-m\mu} \left| \frac{B_\lambda^{k-1}(x)}{x^k} \right|^m dx = o(X^{1-m\mu}).$$

PROOF. It follows by integration by parts.

5. Proof of the theorem. Let us suppose without loss of generality $s = 0$. The necessity part follows from Lemma 2. For sufficiency, it is required to prove, by Lemma 3 that (1.3) with $s = 0$ and (3.1) together imply

$$\int_0^X |c_\lambda^k(x)|^m\,dx = o(X), \quad \text{as } X \to \infty. \tag{5.1}$$

We take throughout $R = \frac{1}{X}$ so that as $R \to 0+$, we have

$$\frac{1}{e}\int_0^X |c_\lambda^k(x)|^m\,dx \leqslant \int_R^\infty dt \int_0^X |c_\lambda^k(x)|^m x e^{-tx}\,dx.$$

Hence by Lemma 3, if $m > 1$ and trivially if $m = 1$, the conclusion will follow if we prove that

$$\int_R^\infty |\phi(t)|^m\,dt \int_0^X x e^{-tx}\,dx = o(X); \tag{5.2}$$

$$\int_R^\infty |\phi(t) - c_\lambda^k(x)|^m \int_0^X x e^{-tx}\,dx = o(X). \tag{5.3}$$

Since the inner integral in (5.2) is less than $\int_0^\infty x e^{-tx}\,dx = 1/t^2$, (5.2) follows at once from (1.3). So we have to consider only (5.3). Now by Lemma 1 and Hölder's inequality,

$$\begin{aligned} |c_\lambda^k(x) - \phi(t)|^m &\leqslant M t^{k+1}\left(\int_0^x v^k |c_\lambda^k(x) - c_\lambda^k(v)|^m e^{-tv}\,dv\right) \\ &\quad + M t^{k+1}\left(\int_x^\infty v^k |c_\lambda^k(x) - c_\lambda^k(v)|^m e^{-tv}\,dv\right) \qquad (5.4) \\ &= S(x, t) + T(x, t). \end{aligned}$$

If $v < x$, we have (by Hölder's inequality when $m > 1$ and trivially when $m = 1$)

$$|c_\lambda^k(x) - c_\lambda^k(v)|^m = M\left|\int_v^x \frac{B_\lambda^{k-1}(u)}{u^{k+1}}\,du\right|^m \leqslant \frac{M}{v}\int_v^x \left|\frac{B_\lambda^{k-1}(u)}{u^k}\right|^m du. \tag{5.5}$$

Since $k > 0$, using (5.5) we have

$$S(x, t) \leqslant M t \int_0^X \left|\frac{B_\lambda^{k-1}(u)}{u^k}\right|^m du = o(tx)$$

uniformly in t by (3.1).

We consider now the contribution of the term $S(x, t)$ to the integral on the left of (5.3); that is to say,

$$\int_R^\infty dt \int_0^X x e^{-tx} S(x, t)\,dx. \tag{5.6}$$

Given ε, there is an x_0 such that, for $x > x_0$ and all $t > 0$, $S(x, t) < \varepsilon x t$. Hence the contribution of $S(x, t)$ to (5.6) if $x > x_0$ is less than

$$\begin{aligned} \varepsilon\int_R^\infty t\,dt \int_{x_0}^X x^2 e^{-tx}\,dx &< \varepsilon\int_0^X x^2\,dx \int_R^\infty t e^{-tx}\,dt \\ &= \int_0^X (1 + Rx)e^{-Rx}\,dx = D\varepsilon X, \end{aligned}$$

where D is a constant. If $\lambda_0 > 0$ we have $B_\lambda^{k-1}(u) = 0$ for $u < \lambda_0$. If $\lambda_0 = 0$ then the term corresponding to $n = 0$ in the sum defining $B_\lambda^{k-1}(u)$ vanishes and hence $B_\lambda^{k-1}(u) = 0$ for $u < \lambda_1$. Thus, in any case, there is some $\eta > 0$ such that $B_\lambda^{k-1}(u) = 0$ for $u < \eta$. Thus we have

$$S(x, t) \leqslant Mt\int_0^x \left|\frac{B_\lambda^{k-1}(u)}{u^k}\right|^m du = 0 \quad \text{for } x < \eta \text{ (whatever } t\text{)}.$$

Hence the integral $\int_R^\infty dt \int_0^{x_0} xe^{-tx}S(x, t)\, dx$ may be reported by

$$\int_R^\infty t\, dt \int_\eta^{x_0} xe^{-tx}\, dx.$$

Once x_0 has been fixed, the inner integral is less than or equal to $e^{-t\eta}\int_\eta^{x_0} x\, dx = ce^{-t\eta}$ (c is a constant) and this gives the required result.

Let μ be a number such that $0 < m\mu < 1$ and also $m\mu < k$. Then if $m > 1$, we have $m'(1 - \mu) > 1$, where $\frac{1}{m} + \frac{1}{m'} = 1$. Now if $v > x$, then, by Hölder's inequality when $m > 1$, and trivially when $m = 1$, we have

$$\tag{5.7} \begin{aligned} |c_\lambda^k(x) - c_\lambda^k(v)|^m &= M\left|\int_x^v \frac{B_\lambda^{k-1}(u)}{u^{k+1}}\, du\right|^m \\ &\leqslant Mx^{m\mu-1}\int_x^v u^{-m\mu}\left|\frac{B_\lambda^{k-1}(u)}{u^k}\right|^m du. \end{aligned}$$

By (5.7) and Lemma 4,

$$\tag{5.8} \begin{aligned} T(x, t) &\leqslant Mt^k x^{m\mu-1}\int_x^\infty u^{-m\mu+k}e^{-tu}\left|\frac{B_\lambda^{k-1}(u)}{u^k}\right|^m du \\ &\quad + Mx^{m\mu-1}\int_x^\infty u^{-m\mu}e^{-tu}\left|\frac{B_\lambda^{k-1}(u)}{u^k}\right|^m du \\ &= U(x, t) + V(x, t). \end{aligned}$$

Now

$$\tag{5.9} \begin{aligned} &\int_R^\infty dt \int_0^X xV(x, t)e^{-tx}\, dx \\ &\quad = M\int_R^\infty dt \int_0^X x^{m\mu}e^{-tx}\, dx\left\{\int_x^X + \int_X^\infty\right\}u^{-m\mu}e^{-tu}\left|\frac{B_\lambda^{k-1}(u)}{u^k}\right|^m du \\ &\quad = J_1 + J_2. \end{aligned}$$

By changing the order of integration

$$\tag{5.10} \begin{aligned} J_1 &\leqslant M\int_R^\infty dt \int_0^X u^{-m\mu}\left|\frac{B_\lambda^{k-1}(u)}{u^k}\right|^m du\int_0^\infty x^{m\mu}e^{-tx}\, dx \\ &= o(1)\left[\frac{X^{1-m\mu}}{R^{m\mu}}\right] = o(X), \end{aligned}$$

by Lemma 5.

The inner integral in J_2, on integration by parts gives in the first place

$$o(X^{1-m\mu}e^{-tX}) + o\left\{\int_X^\infty u|d_u(e^{-tu}u^{-m\mu})|\right\}.$$

Since $e^{-tu}u^{-m\mu}$ is decreasing, we may omit the modulus in the integral if we put a $-$ sign in front; another integration by parts now gives

$$o(X^{1-m\mu}e^{-tX}) + o\left\{\int_X^\infty e^{-tu}e^{-m\mu}\,du\right\}.$$

Again by (4.3) the second term is $o(X^{-m\mu}e^{-tX}/t)$. Since $tX > 1$, this may be absorbed in the first term. Hence

$$J_2 = o(X). \tag{5.11}$$

$$\int_R^\infty dt \int_0^X x \cup (x, t)e^{-tx}\,dx$$

$$\leqslant M\int_R^\infty t^k\,dt\int_0^X x^{m\mu}e^{-tx}\,dx\left(\int_x^X + \int_X^\infty\right)u^{-m\mu+k}e^{-tu}\left|\frac{B_\lambda^{k-1}(u)}{u^k}\right|^m du \tag{5.12}$$

$$= I_1 + I_2.$$

$$\begin{aligned} I_1 &\leqslant \int_R^\infty t^k\,dt\int_0^X u^{k-m\mu}e^{-tu}\left|\frac{B_\lambda^{k-1}(u)}{u^k}\right|^m du\int_0^\infty x^{m\mu}e^{-tx}\,dx \\ &\leqslant M\int_0^X u^{k-m\mu}\left|\frac{B_\lambda^{k-1}(u)}{u^k}\right|^m du\int_R^\infty t^{k-m\mu-1}e^{-tu}\,dt. \end{aligned} \tag{5.13}$$

But since $0 < m\mu < 1$, the inner integral is less than

$$\int_0^\infty t^{k-m\mu-1}e^{-tu}\,dt = \frac{\Gamma(k-m\mu)}{u^{k-m\mu}}$$

and this yields $I_1 = o(X)$.

Lastly, integrating by parts the inner integral in I_2, we get

$$o(X^{1+k-m\mu}e^{-tX}) + o\left\{\int_X^\infty u|d_u(u^{k-m\mu}e^{-tu})|\right\}. \tag{5.14}$$

The expression in curly brackets in (5.14) is equal to

$$-\int_X^\infty u\,d_u(u^{k-m\mu}e^{-tu}) + 2\int_X^{(k-m\mu)/t} u\,d_u(u^{k-m\mu}e^{-tu}),$$

where the second term is to be omitted in the case $tX > k - m\mu$. The first term may be dealt with in the same way as the corresponding term in the treatment of J_2, and the second term may be estimated by noting that, uniformly in the range of integration, $u = O(\frac{1}{t})$. We find that the expression in (5.14) is $o(X^{1+k-m\mu}e^{-tX})$.

Thus we now get

$$
\begin{aligned}
I_2 &= o\left\{X^{1+k-m\mu}\int_R^\infty t^k e^{-tX}\,dt \int_0^X x^{m\mu} e^{-tx}\,dx\right\} \\
&= o\left\{X^{1+k-m\mu}\int_0^\infty t^k e^{-tX}\,dt \int_0^\infty x^{m\mu} e^{-tx}\,dx\right\} \qquad (5.15)\\
&= o\left\{X^{1+k-m\mu}\int_0^\infty t^{k-m\mu-1} e^{-tX}\,dt\right\} \\
&= o(X),
\end{aligned}
$$

since $k > m\mu$. Combining all these (5.4) to (5.15) we get (5.3).

This completes the proof of the theorem.

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