PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 84, Number 2, February 1982

SUBSPACE MAPS OF OPERATORS ON HILBERT SPACE

W. E. LONGSTAFF

ABSTRACT. An operator A acting on a Hilbert space H gives rise to a map φ_A on the set of subspaces of H given by $\varphi_A(M) = \overline{AM}$, where '$^-$' denotes norm closure. This map is called the *subspace map of A*. By identifying subspaces with projections in the usual way it is shown that for $A \neq 0$, φ_A is uniformly (respectively, strongly) continuous if and only if the approximate point spectrum of A does not contain 0. In this case it is proved that φ_A preserves the property of being uniformly (respectively, strongly, weakly) closed and its effect on reflexivity is described.

1. Introduction. Let H be a complex nonzero Hilbert space. Denote by $\mathfrak{B}(H)$ the set of (bounded linear) operators acting on H and denote the set of (closed linear) subspaces of H by $\mathcal{C}(H)$. An operator $A \in \mathfrak{B}(H)$ gives rise to a map φ_A: $\mathcal{C}(H) \to \mathcal{C}(H)$ given by $\varphi_A(M) = \overline{AM}$, where '$^-$' denotes norm closure. We call this map the *subspace map of A*. What are the properties of subspace maps? This is not an easy question to answer especially if one requires the answer to include a resolution of the most famous unsolved problem concerning invariant subspaces, namely, is the statement 'for every $A \in \mathfrak{B}(H)$ there exists $M \in \mathcal{C}(H)$ with $M \neq (0)$, H such that $\varphi_A(M) \subseteq M$' true or false. A lattice-theoretic property of φ_A is that it is a residuated map on the complete lattice $\mathcal{C}(H)$ [1]. This is equivalent to saying that φ_A is a complete join-homomorphism on $\mathcal{C}(H)$, that is, the equality

$$\varphi_A(\bigvee M_\alpha) = \bigvee \varphi_A(M_\alpha)$$

holds for arbitrary families $\{M_\alpha\}$ of subspaces of H, where '$\bigvee$' denotes 'closed linear span'.

The present work concerns topological properties of subspace maps and continues, to the case of not necessarily invertible operators, an investigation begun in [5]. By identifying subspaces and (orthogonal) projections in the usual way, the uniform and strong operator topologies induce topologies on $\mathcal{C}(H)$. In §3 (Theorem 1) it is shown that for $A \neq 0$, φ_A is uniformly (respectively, strongly) continuous if and only if the approximate point spectrum of A does not contain 0. In §4 it is shown that if the approximate point spectrum of A does not contain 0, then φ_A preserves the property of being uniformly (respectively, strongly) closed. Since in this case φ_A is injective, it is a homeomorphism onto its range. We conclude by showing that, for such A, the image $\mathcal{G}$ under φ_A of a reflexive family $\mathcal{F}$ of subspaces is almost reflexive in the sense that $\mathcal{G} \cup \{H\}$ is reflexive.

Received by the editors January 23, 1981.
AMS (MOS) subject classifications (1970). Primary 47B99.

2. Preliminaries. Throughout this paper H will denote a complex nonzero Hilbert space. The inner-product on H is denoted by $(\cdot \mid \cdot)$. For $x \in H$, $\langle x \rangle$ denotes the subspace spanned by x. If $e, f \in H$, $e \otimes f$ denotes the operator defined on H by $(e \otimes f)x = (x|e)f$. For $A \in \mathfrak{B}(H)$, $\mathfrak{R}(A)$ denotes the range of A. For $M \in \mathcal{C}(H)$, P_M denotes the (orthogonal) projection with range M. For $e \in H$, $\|e\| = 1$ we have $P_{\langle e \rangle} = e \otimes e$. The map $M \to P_M$ is a bijection of $\mathcal{C}(H)$ onto the set of projection operators on H. Using this correspondence one can topologize $\mathcal{C}(H)$ by inducing topologies on $\mathfrak{B}(H)$ onto the set of projections. We are concerned with the topologies on $\mathcal{C}(H)$ obtained in this way from the uniform and from the strong operator topologies. Most of the results, notation and terminology we use concerning Hilbert spaces can be found in almost any text on operator theory, for example [2]. If $A \in \mathfrak{B}(H)$ and $M \in \mathcal{C}(H)$, A is said to be *bounded below on* M if there exists $\varepsilon > 0$ such that $\|Ax\| \geqslant \varepsilon\|x\|$ for every $x \in M$. The *approximate point spectrum* $\pi(A)$ of A is the set of scalars λ for which there exists a sequence $\{x_n\}$ of unit vectors of H such that $\{\|Ax_n - \lambda x_n\|\}$ converges to 0. For an operator $A \in \mathfrak{B}(H)$ the following are equivalent (see [2, p. 206 and p. 241ff.])

(1) A is bounded below on H,

(2) $0 \notin \pi(A)$,

(3) A is injective with closed range,

(4) A^* is surjective.

An operator $A \in \mathfrak{B}(H)$ is a *contraction* if $\|A\| \leqslant 1$. If $\mathcal{A}$ is a family of operators on H and $\mathcal{F}$ is a family of subspaces of H, as in [4], we let Lat $\mathcal{A}$ be the set of subspaces of H invariant under every member of $\mathcal{A}$ and let Alg $\mathcal{F}$ be the set of operators on H leaving every member of $\mathcal{F}$ invariant. $\mathcal{F}$ is called *reflexive* if $\mathcal{F} = \text{Lat Alg } \mathcal{F}$.

An abstract lattice L is called *complete* if every family of elements of L has a join and a meet. A subset of a complete lattice L is called a *complete sublattice* of L if it is closed under the formation of arbitrary joins and arbitrary meets. If L_1 and L_2 are complete lattices, an *order isomorphism of* L_1 *onto* L_2 is a bijection $\varphi: L_1 \to L_2$ with the property that $a \leqslant b$ if and only if $\varphi(a) \leqslant \varphi(b)$. A *complete isomorphism of* L_1 *onto* L_2 is a bijection $\psi: L_1 \to L_2$ satisfying $\psi(\bigvee a_\alpha) = \bigvee \psi(a_\alpha)$ and $\psi(\bigwedge a_\alpha) = \bigwedge \psi(a_\alpha)$ identically. It is readily shown that a map of L_1 into L_2 is an order isomorphism if and only if it is a complete isomorphism. In any partially ordered set P, if $a, b \in P$, $[a, b]$ denotes the subset $\{c \in P: a \leqslant c \leqslant b\}$.

3. Continuity of subspace maps. We begin by obtaining an expression for the projection with range $\varphi_A(M)$, for certain operators A and subspaces M, in terms of A and P_M.

PROPOSITION 1. *Let* $A \in \mathfrak{B}(H)$ *be a contraction which is bounded below on the subspace* M. *Then*

(i) AM *is closed,*

(ii) $\|P_M(1 - A^*A)\| < 1$,

(iii) $1 - P_M(1 - A^*A)$ *is invertible with*

$$\|[1 - P_M(1 - A^*A)]^{-1}\| \leq (1 - \|P_M(1 - A^*A)\|)^{-1},$$

(iv) $P_{AM} = A[1 - P_M(1 - A^*A)]^{-1}P_MA^*$.

PROOF. If $M = (0)$ the results follow trivially. Suppose $M \neq (0)$ and let ε be a positive real number such that $\|Aw\| \geq \varepsilon\|w\|$ for every $w \in M$. Since A is a contraction, $\varepsilon \leq 1$. It is clear that AM is closed. Since A is a contraction $1 - A^*A \geq 0$ and so $\|1 - A^*A\| = \sup_{\|u\|=1}(u - A^*Au|u) \leq 1$. If $(1 - A^*A)^{1/2}$ denotes the positive square root of $1 - A^*A$ we have $\|(1 - A^*A)^{1/2}\|^2 = \|1 - A^*A\|$ so $\|(1 - A^*A)^{1/2}\| \leq 1$. Let $v \in H$ satisfy $\|v\| \leq 1$ and put $w = P_Mv$. Then

$$\begin{aligned}\|(1 - A^*A)^{1/2}w\|^2 &= ((1 - A^*A)w|w) \\ &= \|w\|^2 - \|Aw\|^2 \leq (1 - \varepsilon^2)\|w\|^2 \leq 1 - \varepsilon^2.\end{aligned}$$

Hence $\|(1 - A^*A)^{1/2}P_M\| \leq (1 - \varepsilon^2)^{1/2}$ and so

$$\begin{aligned}\|P_M(1 - A^*A)\| &= \|(1 - A^*A)P_M\| \\ &\leq \|(1 - A^*A)^{1/2}\|\,\|(1 - A^*A)^{1/2}P_M\| \leq (1 - \varepsilon^2)^{1/2} < 1.\end{aligned}$$

This proves (ii) and (iii) follows by a well-known result. Finally, let $x \in H$ and put $y = P_{AM}x$. Then $y = Az$ for some $z \in M$ and $x - y \in (AM)^\perp$. Since

$$A^*(AM)^\perp \subseteq M^\perp, \quad A^*(x - y) \in M^\perp$$

so

$$P_MA^*x = P_MA^*y.$$

Hence $P_MA^*x = [1 - P_M(1 - A^*A)]z$ so $z = [1 - P_M(1 - A^*A)]^{-1}P_MA^*x$. Thus $P_{AM}x = y = A[1 - P_M(1 - A^*A)]^{-1}P_MA^*x$ and (iv) follows. This completes the proof.

COROLLARY 1. *Let A and M be as in the above proposition. If the net $\{K_\alpha\}$ of subspaces converges uniformly (respectively, strongly) to the subspace K and $K_\alpha \subseteq M$ for every α, then $K \subseteq M$, AK and AK_α are closed and $\{AK_\alpha\}$ converges uniformly (respectively, strongly) to AK.*

PROOF. For every α, $P_MP_{K_\alpha} = P_{K_\alpha}$. Since $\{P_MP_{K_\alpha}\}$ converges uniformly (respectively, strongly) to P_MP_K, $P_MP_K = P_K$ so $K \subseteq M$. Since A is bounded below on K_α and on K, AK_α and AK are closed and $P_{AK_\alpha} = AT_\alpha^{-1}P_{K_\alpha}A^*$, $P_{AK} = AT^{-1}P_KA^*$ where $T_\alpha = 1 - P_{K_\alpha}(1 - A^*A)$ and $T = 1 - P_K(1 - A^*A)$. Clearly $\{T_\alpha\}$ converges uniformly (respectively, strongly) to T. Since

$$\|T_\alpha^{-1}\| \leq \left(1 - \|P_{K_\alpha}(1 - A^*A)\|\right)^{-1} \leq (1 - \|P_M(1 - A^*A)\|)^{-1},$$

$\{\|T_\alpha^{-1}\|\}$ is bounded. From this and the equalities $T^{-1} - T_\alpha^{-1} = T_\alpha^{-1}(T_\alpha - T)T^{-1}$ it follows that $\{T_\alpha^{-1}\}$ converges uniformly (respectively, strongly) to T^{-1}. An easy argument now shows that $\{P_{AK_\alpha}\}$ converges uniformly (respectively, strongly) to P_{AK}. This completes the proof.

For the above corollary, a proof of the uniform convergence of $\{AK_\alpha\}$ to AK follows almost immediately from Lemma 1.1 of [3].

COROLLARY 2. *If $S \in \mathfrak{B}(H)$ is an invertible contraction and M is a subspace of H, the operator $T_M = 1 + SP_M(S^* - S^{-1})$ is invertible with inverse $1 - P_{SM} + P_{SM}S^{*-1}S^{-1}$. Moreover, $P_{SM} = T_M^{-1}SP_MS^*$ and $\|T_M^{-1}\| \leq 1 + \|S^{*-1}S^{-1}\|$.*

PROOF. Since S is bounded below on M, $1 - P_M(1 - S^*S)$ is invertible. We have $T_M = S[1 - P_M(1 - S^*S)]S^{-1}$ so T_M is invertible. Also,

$$P_{SM} = S[1 - P_M(1 - S^*S)]^{-1}P_MS^* = SS^{-1}T_M^{-1}SP_MS^* = T_M^{-1}SP_MS^*$$

and so

$$\begin{aligned} 1 - P_{SM} + P_{SM}S^{*-1}S^{-1} &= 1 - T_M^{-1}SP_MS^* + T_M^{-1}SP_MS^{-1} \\ &= T_M^{-1}\big(T_M - SP_M(S^* - S^{-1})\big) = T_M^{-1}. \end{aligned}$$

That $\|T_M^{-1}\| \leq 1 + \|S^{*-1}S^{-1}\|$ readily follows. This completes the proof.

In the above corollary the requirement that S be a contraction may be dropped [5, Lemma]. For an arbitrary invertible S, by applying the corollary to the invertible contraction $A = S/\|S\|$, noting that $P_{AM} = P_{SM}$ and performing some manipulations (possibly involving $P_{SM}S^{*-1}(1 - P_M) = P_MS^*(1 - P_{SM}) = 0$), the corresponding results for S can be obtained.

We can now prove the main result of this section.

THEOREM 1. *Let $A \in \mathfrak{B}(H)$ with $A \neq 0$. The subspace map φ_A: $\mathcal{C}(H) \to \mathcal{C}(H)$ of A is uniformly (respectively, strongly) continuous if and only if $0 \notin \pi(A)$.*

PROOF. If $0 \notin \pi(A)$, then A is bounded below on H and so is the contraction $A/\|A\|$. The uniform (respectively, strong) continuity of φ_A follows from Corollary 1 and the observation that $\varphi_A = \varphi_{A/\|A\|}$.

Suppose that φ_A is uniformly (respectively, strongly) continuous. We show that A is injective. Suppose not. Let $e \in \ker A$ with $\|e\| = 1$ and let $f \in (\ker A)^\perp$ with $\|f\| = 1$. For each positive integer n put $e_n = \cos \frac{1}{n}e + \sin \frac{1}{n}f$. Then $\|e_n\| = 1$ and $\{e_n\}$ converges to e. For each $g \in H$ we have

$$\begin{aligned} \|(P_{\langle e_n\rangle} - P_{\langle e\rangle})g\| &= \|(g|e_n)e_n - (g|e)e\| \\ &= \|(g|e_n - e)e_n + (g|e)(e_n - e)\| \\ &\leq 2\|e_n - e\|\,\|g\|. \end{aligned}$$

It follows that the sequence of subspaces $\{\langle e_n\rangle\}$ converges uniformly to $\langle e\rangle$. Since $A\langle e_n\rangle = \langle Af\rangle \neq (0)$, $\{A\langle e_n\rangle\}$ does not converge strongly to $A\langle e\rangle$. This contradiction proves that A is injective.

Now suppose that φ_A is uniformly continuous but that $0 \in \pi(A)$. Then A^* is not surjective. Since A is injective, $\mathfrak{R}(A^*)$ is dense so $\mathfrak{R}(A^*)$ is not closed. Let $x \in \overline{\mathfrak{R}(A^*)} \setminus \mathfrak{R}(A^*)$ with $\|x\| = 1$. There is a sequence $\{x_n\}$ of unit vectors, each belonging to $\mathfrak{R}(A^*)$, converging to x. (If $z_n \in \mathfrak{R}(A^*)$ and $\{z_n\}$ converges to x, we may suppose $\|z_n\| \geq \frac{1}{2}$. Then $x_n = z_n/\|z_n\|$ gives such a sequence.) Put $M_n = \langle x_n\rangle^\perp$ and $M = \langle x\rangle^\perp$. Then $\{M_n\}$ converges uniformly to M. Now $\overline{AM} = \overline{\mathfrak{R}(AP_M)} = (\ker P_MA^*)^\perp$. Since $P_M = 1 - x \otimes x$, $\ker P_MA^* = \{y \in H: A^*y = (A^*y|x)x\}$. Since $x \notin \mathfrak{R}(A^*)$, $\ker P_MA^* = \ker A^*$ and so $\overline{AM} = \overline{\mathfrak{R}(A)}$.

We also have $\overline{AM_n} = (\ker P_{M_n}A^*)^\perp$ and $\ker P_{M_n}A^* = \{y \in H: A^*y = (A^*y|x_n)x_n\}$. Since $x_n \in \mathfrak{R}(A^*)$, $x_n = A^*y_n$ for some $y_n \in H$. Clearly $y_n \in \ker P_{M_n}A^*$ and $y_n \notin \ker A^*$. Hence $\ker A^* \subset \ker P_{M_n}A^*$ (strict inclusion) and so, taking orthocomplements, $\overline{AM_n} \subset \overline{AM}$. Thus $\|P_{\overline{AM_n}} - P_{\overline{AM}}\| = 1$ for every n so $\{\overline{AM_n}\}$ does not converge uniformly to $\overline{AM}$. This contradiction proves that $0 \notin \pi(A)$.

Finally, suppose that φ_A is strongly continuous. Then A is injective. It is easily shown that a sequence $\{w_n\}$ of unit vectors of H converges weakly to 0 if and only if the sequence of subspaces $\{\langle w_n \rangle\}$ converges strongly to (0). It follows that A has the following property: for every sequence $\{w_n\}$ of unit vectors converging weakly to 0, the sequence $\{Aw_n/\|Aw_n\|\}$ converges weakly to 0. The proof of the theorem is completed by showing that every injective operator $B \in \mathfrak{B}(H)$ with this property has closed range and so $0 \notin \pi(B)$.

Suppose $\mathfrak{R}(B)$ is not closed. Let $v \in \overline{\mathfrak{R}(B)} \setminus \mathfrak{R}(B)$ with $\|v\| = 1$. There is a sequence $\{v_n\}$ of unit vectors, each belonging to $\mathfrak{R}(B)$, converging to v. Now $v_n = Bw_n$ for some $w_n \neq 0$. If $\{w_n\}$ possessed a (norm) bounded infinite subsequence $\{w_{n_j}\}$ say, this subsequence would, in turn, possess a weakly convergent subsequence $\{w'_{n_j}\}$ say, converging weakly to, say, w. Then $\{Bw'_{n_j}\}$ would converge weakly to Bw. But $\{Bw'_{n_j}\}$ converges to v so we would have $v = Bw$. Since $v \notin \mathfrak{R}(B)$, no infinite subsequence of $\{w_n\}$ is bounded and so every infinite subsequence of $\{\|w_n\|\}$ converges to ∞. Consider the sequence $\{w_n/\|w_n\|\}$. This bounded sequence possesses a weakly convergent infinite subsequence $\{w_{n_j}/\|w_{n_j}\|\}$ say. Let the weak limit be u. Then $\{Bw_{n_j}/\|w_{n_j}\|\}$ converges weakly to Bu. Since $\|Bw_{n_j}\| = \|v_{n_j}\| = 1$ and $\{\|w_{n_j}\|\}$ converges to ∞, $\{Bw_{n_j}/\|w_{n_j}\|\}$ converges to 0. Thus $Bu = 0$ so $u = 0$. By the property of B, since $\{w_{n_j}/\|w_{n_j}\|\}$ converges weakly to 0, $\{Bw_{n_j}/\|Bw_{n_j}\|\}$ converges weakly to 0. That is, $\{v_{n_j}\}$ converges weakly to 0. But $\{v_{n_j}\}$ converges to v so $v = 0$. This contradicts $v \notin \mathfrak{R}(B)$. Thus B has closed range and the proof of the theorem is complete.

REMARKS. 1. It may be of some independent interest to note that, for an injective operator B, $0 \notin \pi(B)$ if and only if for every sequence $\{w_n\}$ of unit vectors converging weakly to 0 the sequence $\{Bw_n/\|Bw_n\|\}$ converges weakly to 0. The necessity of the condition is easily established, the sufficiency is proved above.

2. The above proof also shows that, for an injective operator $A \in \mathfrak{B}(H)$, φ_A is strongly continuous if and only if for every sequence $\{M_n\}$ of *one-dimensional* subspaces converging strongly to (0) the sequence $\{AM_n\}$ converges strongly to (0). The latter condition does not imply that A is injective even if $A \neq 0$. (For example, consider $A = 1 - e \otimes e$ where $e \in H$ with $\|e\| = 1$.)

3. If H is infinite-dimensional and K is a nonzero compact operator on H, then φ_K is neither uniformly nor strongly continuous (since $0 \in \pi(K)$).

4. For a nonzero normal operator $A \in \mathfrak{B}(H)$, φ_A is uniformly (respectively, strongly) continuous if and only if A is invertible (since $\pi(A) = \sigma(A)$, where $\sigma(A)$ denotes the spectrum of A).

5. For a nonzero operator $A \in \mathfrak{B}(H)$, both φ_A and φ_{A^*} are uniformly (respectively, strongly) continuous if and only if A is invertible. The proof is immediate.

4. Transforms of families of subspaces. Let $A \in \mathfrak{B}(H)$ and let $\mathfrak{F}$ be a family of subspaces of H. Call the family of subspaces $\{\varphi_A(M): M \in \mathfrak{F}\}$ the *transform of* $\mathfrak{F}$ *by* A. This extends the definition given in [5]. As in [5], call $\mathfrak{F}$ uniformly (respectively, strongly, weakly) closed if the set of projections $\{P_M: M \in \mathfrak{F}\}$ is a uniformly (respectively, strongly, weakly) closed subset of $\mathfrak{B}(H)$. If $0 \notin \pi(A)$ the subspace map $\varphi_A: \mathcal{C}(H) \to \mathcal{C}(H)$ is given by $\varphi_A(M) = AM$. Since, in this case, A is injective so is φ_A. A description of the range of φ_A and its inverse map follows.

LEMMA. *Let $A \in \mathfrak{B}(H)$ with $0 \notin \pi(A)$. The range of the subspace map φ_A is $\{M \in \mathcal{C}(H): M \subseteq \mathfrak{R}(A)\}$. For every subspace M in the range of φ_A, $A^*M^\perp$ is closed and $\varphi_A^{-1}(M) = (A^*M^\perp)^\perp$.*

PROOF. It is clear that the range of φ_A is included in $\{M \in \mathcal{C}(H): M \subseteq \mathfrak{R}(A)\}$. Let $M \in \mathcal{C}(H)$ with $M \subseteq \mathfrak{R}(A)$ and put $N = (A^*M^\perp)^\perp$. We show that $M = \varphi_A(N)$. For every $x \in N$ and $y \in M^\perp$, $(Ax|y) = (x|A^*y) = 0$ so $\varphi_A(N) \subseteq M$. Let $v \in M$. Then $v = Aw$ for some $w \in H$. For every $u \in M^\perp$, $(w|A^*u) = (Aw|u) = (v|u) = 0$ so $w \in N$ and $v \in \varphi_A(N)$. Hence $M = \varphi_A(N)$. Finally, we show $A^*M^\perp = N^\perp$. Clearly $A^*M^\perp \subseteq N^\perp$. Let $s \in N^\perp$. Since A^* is surjective $s = A^*t$ for some $t \in H$. For every $r \in N$, $(t|Ar) = (A^*t|r) = (s|r) = 0$ so $t \in (\varphi_A(N))^\perp = M^\perp$ and $s \in A^*M^\perp$. Thus $A^*M^\perp = N^\perp$ and the proof is complete.

We can now prove the main result of this section.

THEOREM 2. *Let $A \in \mathfrak{B}(H)$ with $0 \notin \pi(A)$. Let $\mathfrak{F}$ be a family of subspaces of H and let $\mathcal{G}$ be the transform of $\mathfrak{F}$ by A. If $\mathfrak{F}$ is uniformly (respectively, strongly, weakly) closed so is $\mathcal{G}$.*

PROOF. Let $\mathfrak{F}$ be uniformly (respectively, strongly) closed and let $\{P_{AM_\alpha}\}$ be a net converging uniformly (respectively, strongly) to the (necessarily a projection) operator P_N with $M_\alpha \in \mathfrak{F}$ for every α. It is clear that $N \subseteq \mathfrak{R}(A)$ (cf. Corollary 1) so, by the lemma, $N = AM$ for some subspace M. We need to show that $M \in \mathfrak{F}$. For every α we have $\ker A^* \subseteq (AM_\alpha)^\perp$ and $\ker A^* \subseteq (AM)^\perp$. Put $K_\alpha = (AM_\alpha)^\perp \ominus \ker A^*$ and $K = (AM)^\perp \ominus \ker A^*$. Then $K_\alpha \subseteq \mathfrak{R}(A)$ and $K \subseteq \mathfrak{R}(A)$. Now A^* maps $\mathfrak{R}(A)$ bijectively onto H and so is bounded below on $\mathfrak{R}(A)$. Since $\{K_\alpha\}$ converges uniformly (respectively, strongly) to K, $\{A^*K_\alpha\}$ converges uniformly (respectively, strongly) to A^*K by Corollary 1. But $A^*K_\alpha = A^*(AM_\alpha)^\perp = M_\alpha^\perp$ and $A^*K = A^*(AM)^\perp = M^\perp$ by the lemma. Thus $\{M_\alpha\}$ converges uniformly (respectively, strongly) to M so $M \in \mathfrak{F}$.

Finally, suppose $\mathfrak{F}$ is weakly closed. Let the operator $E \in \mathfrak{B}(H)$ be the weak limit of a net $\{P_{AM_\alpha}\}$ with $M_\alpha \in \mathfrak{F}$ for every α. By the weak compactness of the unit ball of $\mathfrak{B}(H)$ the net $\{P_{M_\alpha}\}$ possesses a subnet converging weakly to an operator $F \in \mathfrak{B}(H)$. Since $\mathfrak{F}$ is weakly closed $F = P_M$ for some $M \in \mathfrak{F}$. This subnet converges strongly to P_M so P_{AM} is the strong limit of a subnet of $\{P_{AM_\alpha}\}$. But this subnet converges weakly to E so $E = P_{AM}$. This proves that $\mathcal{G}$ is weakly closed and the proof is complete.

Let $A \in \mathfrak{B}(H)$ with $0 \notin \pi(A)$. The subspace map φ_A is an order isomorphism of $\mathcal{C}(H)$ onto its complete sublattice $[(0), \mathfrak{R}(A)]$ and so is a complete isomorphism of

$\mathcal{C}(H)$ onto $[(0), \mathfrak{R}(A)]$. Also, Theorems 1 and 2 show that φ_A is a homeomorphism of $\mathcal{C}(H)$ onto $[(0), \mathfrak{R}(A)]$ if each is furnished with either the induced uniform or with the induced strong operator topology. If $\mathcal{F}$ is a reflexive family of subspaces of H, then $\mathcal{F}$ is strongly closed [4] and so its transform $\mathcal{G}$ by A is strongly closed. Since a reflexive family necessarily contains H, $\mathcal{G}$ is not reflexive if A is not invertible. However $\mathcal{G} \cup \{H\}$ is always reflexive.

PROPOSITION 2. *Let $A \in \mathfrak{B}(H)$ with $0 \notin \pi(A)$. Let $\mathcal{F}$ be a family of subspaces of H containing H and let $\mathcal{G}$ be the transform of $\mathcal{F}$ by A. Then* Lat Alg $\mathcal{G} = \mathcal{K} \cup \{H\}$ *where $\mathcal{K}$ is the transform of* Lat Alg $\mathcal{F}$ *by A. If $\mathcal{F}$ is reflexive so is $\mathcal{G} \cup \{H\}$.*

PROOF. Let $M \in$ Lat Alg $\mathcal{F}$ and let $T \in$ Alg $\mathcal{G}$. Since $\mathfrak{R}(A) \in \mathcal{G}$, T leaves $\mathfrak{R}(A)$ invariant and so the transformation $X = A^{-1}TA$ is bounded on H. Clearly $X \in$ Alg $\mathcal{F}$ so X leaves M invariant. It follows that T leaves AM invariant. This shows that $\mathcal{K} \cup \{H\} \subseteq$ Lat Alg $\mathcal{G}$.

Now let $N \in$ Lat Alg $\mathcal{G}$. If E denotes the projection with range $\mathfrak{R}(A)$, $B(1 - E) \in$ Alg $\mathcal{G}$ for every operator $B \in \mathfrak{B}(H)$. Thus $B(1 - E)x \in N$ for every $x \in N$ and it follows that either $N \subseteq \mathfrak{R}(A)$ or $N = H$. Suppose $N \neq H$. Then $N = AK$ for some subspace K. We show that $K \in$ Lat Alg $\mathcal{F}$. Let $S \in$ Alg $\mathcal{F}$. The transformation ASA^{-1} is bounded on $\mathfrak{R}(A)$ so the transformation $R = (ASA^{-1})E$ is bounded on H. We have $RA = AS$ and clearly $R \in$ Alg $\mathcal{G}$. Thus R leaves N invariant so $RN = RAK = ASK \subseteq AK$. It follows that K is invariant under S and that $N \in \mathcal{K}$. Hence Lat Alg $\mathcal{G} = \mathcal{K} \cup \{H\}$.

If $\mathcal{F}$ is reflexive we have $\mathcal{K} = \mathcal{G}$ and

$$\text{Lat Alg}(\mathcal{G} \cup \{H\}) = \text{Lat Alg } \mathcal{G} = \mathcal{G} \cup \{H\}$$

so $\mathcal{G} \cup \{H\}$ is reflexive. This completes the proof.

REFERENCES

1. H.-J. Bandelt, *Tight residuated mappings and d-extensions*, (Proc. Colloq. on Universal Algebra, Esztergom, 1977), Colloq. Math. Soc. János Bolyai; Universal Algebra.

2. S. K. Berberian, *Lectures in functional analysis and operator theory*, Springer-Verlag, Berlin and New York, 1974.

3. R. G. Douglas and Carl Pearcy, *On a topology for invariant subspaces*, J. Funct. Anal. **2** (1968), 323–341.

4. P. R. Halmos, *Reflexive lattices of subspaces*, J. London Math. Soc. **4** (1971), 257–263.

5. W. E. Longstaff, *A note on transforms of subspaces of Hilbert space*, Proc. Amer. Math. Soc. **76** (1979), 268–270.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WESTERN AUSTRALIA, NEDLANDS, WESTERN AUSTRALIA 6009, AUSTRALIA