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UNIQUENESS AND QUASI-MEASURES ON THE GROUP OF INTEGERS OF A p-SERIES FIELD

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ABSTRACT. Let G be the group of integers of a p-series field and suppose that S is a character series on G. If $N_1, N_2, \ldots$ is any sequence of integers and if $S_{p^{N_j}} \to 0$ a.e. on G, as $j \to \infty$, then S will be the zero series provided S never diverges unboundedly.

Let G denote the group of integers of a p-series field, where p is a prime $\geqslant 2$. Thus, any element $\bar{x} \in G$ can be represented as a sequence $\{x_i\}_{i=0}^{\infty}$ with $0 \leqslant x_i < p$ for each $i \geqslant 0$. Moreover, the dual group $\{\psi_m\}_{m=0}^{\infty}$ of G can be described by the following process. If m is a nonnegative integer with $m = \sum_{k=0}^{\infty} \alpha_k p^k$, $0 \leqslant \alpha_k < p$ for each k, and if $\bar{x} \in G$ then

$$\psi_m(\bar{x}) = \prod_{k=0}^{\infty} \phi_k^{\alpha_k}(\bar{x}), \tag{1}$$

where for each integer $k \geqslant 0$ and for each $\bar{x} = \{x_i\} \in G$, the function ϕ_k is defined by

$$\phi_k(\bar{x}) = \exp(2\pi i x_k/p). \tag{2}$$

In the case that $p = 2$, the group G is the dyadic group introduced by Fine [2] and the functions $\{\psi_m\}_{m=0}^{\infty}$ are the Walsh-Paley functions. A detailed account of these groups and basic properties can be found in [5].

Denote the partial sums of a character series $S = \sum_{m=0}^{\infty} a_m \psi_m$ by

$$S_N = \sum_{m=0}^{N-1} a_m \psi_m, \qquad N = 1, 2, \ldots. \tag{3}$$

Vilenkin [6] has shown that if $S_N \to 0$ everywhere on G as $N \to \infty$, then S is the zero series, i.e., $a_m = 0$ for $m = 0, 1, \ldots$. When N is replaced by p^N and convergence is relaxed on a countable subset of G, a growth condition is usually necessary to retain uniqueness. For example, in [8] we saw that if S_{p^N} converges to an integrable f on all but countably many points in G, and if $p^{-N} S_{p^N} \to 0$ everywhere on G as $N \to \infty$, then S is the G-Fourier series of f. The second hypothesis of this result cannot be relaxed at a single point $\bar{x}_0 \in G$ where $|S_{p^N}(\bar{x}_0)| \to +\infty$ as

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$N \to \infty$. Indeed, if $D = \sum_{m=0}^{\infty} \psi_m$ represents the Dirichlet kernel on G, then since

$$D_{p^N}(\bar{x}) = \begin{cases} p^N & \text{if } \sum_{i=0}^{\infty} x_i p^{-i-1} < p^{-N}, \\ 0 & \text{otherwise,} \end{cases}$$

it is clear that both $D_{p^N}(\bar{x})$ and $p^{-N} D_{p^N}(\bar{x})$ converge to zero for $\bar{x} \neq 0$ as $N \to \infty$ but D is not the zero series. It is not yet known how far the first hypothesis can be relaxed (see [7]). One would expect that "convergence off a countable set" could be replaced with "convergence a.e." but even in the case $p = 2$ this has not been done.

If one strengthens the second hypothesis to condition (4) below, then convergence a.e. can be used. Indeed, we shall prove the following result.

THEOREM. *Suppose that* $S = \sum_{m=0}^{\infty} a_m \psi_m$ *and that* $\{m_j\}_{j=1}^{\infty}$ *is a subsequence of the natural numbers. If* $S_{p^{m_j}} \to 0$ *a.e. on* G *as* $j \to \infty$, *and if*

$$\limsup_{j \to \infty} |S_{p^{m_j}}(\bar{x})| < \infty, \qquad \bar{x} \in G, \tag{4}$$

then S *is the zero series.*

Techniques used to establish uniqueness for Walsh series fall into two categories: proof by a Haar series argument (e.g., [1]), and proof by differentiation (e.g., [2] and [4]). Neither of these techniques seem suited to prove the theorem above.

Our technique introduces a fresh viewpoint, and uses quasi-measures (defined below) as a crutch for carrying out the necessary calculations. Recall that the topology of G has a base at 0 which consists of closed/open subgroups G_n whose Haar measure $m(G_n)$ equals p^{-n}, $n \geq 0$. We shall denote $G_0 \equiv G$ by $I(0, 0)$, and for each integer $n > 0$ we shall denote the cosets of G_n by $I(k, n)$, $0 \leq k < p^n$. The collection of sets $I(k, n)$, $0 \leq k < p^n$, $n = 0, 1, \ldots,$ will be denoted by $\mathcal{I}$. Observe once and for all that $I(k_1, n) \cap I(k_2, n) = \varnothing$ for $k_1 \neq k_2$, that $m(I(k, n)) = p^{-n}$, that (reordering if necessary)

$$I(k, n) = \bigcup_{l=kp}^{kp+p-1} I(l, n+1) \tag{5}$$

and that each ψ_l is constant on each $I(k, n)$ when $l < p^n$. A set function μ defined on $\mathcal{I}$ is said to be a *quasi-measure* if

$$\mu(I(k, n)) = \sum_{l=kp}^{kp+p-1} \mu(I(l, n+1)). \tag{6}$$

Clearly, every Borel measure on G is also a quasi-measure.

Fix integers k and n with $0 \leq k < p^n$. By an argument similar to that found in [3], one can show that if λ is a Borel measure on G, and if S is its Fourier-Stieltjes series, then

$$\lambda(I(k, n)) = \lim_{N \to \infty} \int_{I(k, n)} S_N \, dm.$$

Since $\int_{I(k,n)} \psi_l \, dm = 0$ for $l \geq p^n$ and since S_{p^n} is constant on $I(k, n)$, it follows that $\lambda(I(k, n)) \equiv p^{-n} S_{p^n}(\bar{x})$ for any choice of $\bar{x} \in I(k, n)$. Thus we are led to associate with any character series S the set function μ defined on $\mathcal{I}$ by

$$\mu(I(k, n)) \equiv p^{-n} S_{p^n}(\bar{x}), \qquad \bar{x} \in I(k, n). \tag{7}$$

A routine calculation (recall that the sum of pth roots of unity is zero) establishes that this set function μ is a quasi-measure. Moreover, since the characters of G are orthogonal it is clear that a necessary and sufficient condition for a character series S to be zero is that its associated quasi-measure satisfies $\mu(I) = 0$ for all $I \in \mathcal{I}$.

We are now prepared to prove the theorem. Indeed, if S is the given series and if μ is its associated quasi-measure, we need only show that $\mu \equiv 0$. We shall actually show that $\mu(G) = 0$. The same argument can be used to show that $\mu(I) = 0$ for all $I \in \mathcal{I}$ and thus complete the proof.

We assume for simplicity that $m_j = j$. Fix $0 < \varepsilon < 1$. By Egoroff's Theorem we can choose a subset E_1 of G such that $m(E_1) > 1 - \varepsilon$ and such that S_{p^N} converges uniformly to zero on E_1, as $N \to \infty$. Thus, given $\varepsilon_1 > 0$ there exists an integer N_1 such that $|S_{p^{N_1}}(\bar{x})| < \varepsilon_1$ for $\bar{x} \in E_1$. But $S_{p^{N_1}}$ is actually constant on sets of the form $I(k, N_1)$ so there exists a subset Z_1 of the positive integers such that $k \in Z_1$ implies $|S_{p^{N_1}}(\bar{x})| < \varepsilon_1$ for $\bar{x} \in I(k, N_1)$.

Observe that if $|Z_1|$ represents the cardinality of Z_1 then

$$|Z_1| \cdot p^{-N_1} \equiv \sum_{k \in Z_1} m(I(k, N_1)) \geqslant m(E_1) > 1 - \varepsilon.$$

Thus $1 - |Z_1| \cdot p^{-N_1} < \varepsilon$. Now, let k_1 be an integer which satisfies

$$|\mu(I(k_1, N_1))| = \max_{k \notin Z_1} |\mu(I(k, N_1))|$$

and observe that $S_{p^N} \to 0$ a.e. on $I(k_1, N_1)$ as $N \to \infty$. Thus, given ε_2 we can choose a subset E_2 of $I(k_1, N_1)$ such that $m(E_2) > (1 - \varepsilon)p^{-N_1}$ and choose an integer N_2 such that $|S_{p^{N_1+N_2}}(\bar{x})| < \varepsilon_2$ for $\bar{x} \in E_2$.

Continuing in this manner, given any positive integer j and $\varepsilon_j > 0$ we can choose positive integers N_j, k_j, subsets Z_j of natural numbers and E_j of G such that

$$1 - |Z_j| p^{-N_j} < \varepsilon, \tag{8}$$

$$\text{if } k \in Z_j \text{ then } |S_{p^{N_1+\cdots+N_j}}(\bar{x})| < \varepsilon_j \quad \text{for } \bar{x} \in I(k, N_1 + \cdots + N_j) \tag{9}$$

and

$$\begin{aligned} &|\mu(I(k_j, N_1 + \cdots + N_j))| \\ &\quad = \max\{|\mu(I(k, N_1 + \cdots + N_j))| : k \notin Z_j \\ &\qquad \text{and } I(k, N_1 + \cdots + N_j) \subseteq I(k_{j-1}, N_1 + \cdots + N_{j-1})\}. \end{aligned} \tag{10}$$

To estimate $\mu(G)$, observe by (6) that $|\mu(G)| \leqslant T_1 + \tilde{T}_1$ where

$$T_1 = \sum\{|\mu(I(k, N_1))| : k \in Z_1\}$$

and

$$\tilde{T}_1 = \sum\{|\mu(I(k, N_1))| : 0 \leqslant k < p^{N_1}, k \notin Z_1\}.$$

To estimate T_1 we observe that $|Z_1|p^{-N_1} \leqslant 1$ follows from the fact that there are at most p^{N_1} sets of the form $I(k, N_1)$. Therefore, by (7) and (9) we have that $T_1 \leqslant \varepsilon_1$.

To estimate $\tilde{T}_1$ we apply (10) to conclude that

$$\tilde{T}_1 \leqslant (p^{N_1} - |Z_1|)|\mu(I(k_1, N_1))|.$$

Thus

$$|\mu(G)| \leqslant \varepsilon_1 + (p^{N_1} - |Z_1|)|\mu(I(k_1, N_1))|.$$

But (6), (7), (9), and (10) can be used to estimate $|\mu(I(k_1, N_1))|$ by summing over those intervals $I(k, N_1 + N_2)$ which are contained in $I(k_1, N_1)$ and separating the indices $k \in Z_2$ from $k \notin Z_2$. Again $|Z_2|p^{-N_2} < 1$ follows from the fact that there are at most p^{N_2} sets of the form $I(k, N_1 + N_2)$ contained in $I(k_1, N_1)$.

If we continue breaking up $|\mu(I(k_j, N_1 + \cdots + N_j))|$ in this manner we arrive at the following inequality:

$$\begin{aligned} |\mu(G)| \leqslant \varepsilon_1 + \varepsilon_2(p^{N_1} - |Z_1|) + \cdots + \varepsilon_j \prod_{l=1}^{j-1} (p^{N_l} - |Z_l|) \\ + |\mu(I(k_j, N_1 + \cdots + N_j))| \cdot \prod_{l=1}^{j} (p^{N_l} - |Z_l|), \end{aligned} \tag{11}$$

$j = 1, 2, \ldots$. Set $\varepsilon_1 = \varepsilon/2$ and for each integer $j > 1$ set

$$\varepsilon_j = 2^{-j}\varepsilon\left(\prod_{l=1}^{j-1} (p^{N_l} - |Z_l|)\right)^{-1}.$$

It follows from (8) and (11) that

$$|\mu(G)| \leqslant \varepsilon(1 - 2^{-j}) + \varepsilon p^{N_1 + \cdots + N_j}|\mu(I(k_j, N_1 + \cdots + N_j))|. \tag{12}$$

Observe by construction that the collection of compact sets

$$\{I(k_j, N_1 + \cdots + N_j)\}_{j=1}^{\infty}$$

is nested. Consesquently, we can choose a point $\bar{x}_0$ which belongs to $I(k_j, N_1 + \cdots + N_j)$ for all integers $j \geqslant 1$. By (7), then, the estimate (12) becomes

$$|\mu(G)| \leqslant \varepsilon + \varepsilon|S_{p^{N_1 + \cdots + N_j}}(\bar{x}_0)|.$$

Our simplifying assumption turns (4) into $\limsup_{N\to\infty} S_{p^N}(\bar{x}_0)| = A < \infty$. Thus

$$|\mu(G)| \leqslant \varepsilon + A\varepsilon.$$

In particular, if we let $\varepsilon \to 0$ we obtain $\mu(G) = 0$, thus completing the proof of the theorem.

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