

A RADON-NIKODYM THEOREM FOR NATURAL CONES ASSOCIATED WITH VON NEUMANN ALGEBRAS

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ABSTRACT. The natural cone associated with a von Neumann algebra admitting a cyclic and separating vector ξ_0 is considered. For any vector ξ in the cone, there always exists a closed operator t affiliated with the algebra satisfying $\xi = tJt\xi_0$.

1. Introduction. Following the development of the Tomita-Takesaki theory [10], a natural cone for a von Neumann algebra was introduced independently by several authors [1, 2, 5] and shown to be a good "invariant" for the algebra (and its commutant) in question. For a von Neumann algebra \mathfrak{M} on a Hilbert space \mathcal{H} with a cyclic and separating vector ξ_0 , the natural cone \mathfrak{P}^h may be defined as the closure of the set of all vectors $aJaJ\xi_0$, $a \in \mathfrak{M}$.

It is thus natural to ask if an arbitrary vector in the cone \mathfrak{P}^h can be expressed as $tJt\xi_0$ with an (unbounded) closed operator t affiliated with \mathfrak{M} . The purpose of the paper is to show the affirmative answer to this question.

2. Notations and the main result. Let $(\mathfrak{M}, \mathcal{H}, \xi_0)$ be as in the introduction and φ_0 be the faithful normal positive functional on \mathfrak{M} defined by $\varphi_0(x) = \omega_{\xi_0}(x) = (x\xi_0|\xi_0)$, $x \in \mathfrak{M}$. We denote the modular operator and the modular conjugation associated with the above triple simply by Δ and J respectively [10]. Fixing these throughout, the modular automorphism group $\{\text{Ad } \Delta^t\}_{t \in \mathbb{R}}$ on \mathfrak{M} will be denoted by $\{\sigma_t\}_{t \in \mathbb{R}}$. Also we denote by \mathfrak{M}_0 (resp. \mathfrak{M}'_0) the set of every $x \in \mathfrak{M}$ (resp. $x' \in \mathfrak{M}'$) such that the map $t \in \mathbb{R} \rightarrow \Delta^t x \Delta^{-t}$ (resp. $t \in \mathbb{R} \rightarrow \Delta^{-t} x' \Delta^t$) extends to an entire function.

DEFINITION 2.1 [1, 2, 5]. The natural cone \mathfrak{P}^h is the closure of $\Delta^{1/4}\mathfrak{M}_+\xi_0$ in \mathcal{H} .

As mentioned earlier, the natural cone \mathfrak{P}^h may be defined as the closure of $\{aJaJ\xi_0; a \in \mathfrak{M}\}$. Among other properties, the cone enjoys the following.

PROPOSITION 2.2 [1, 2, 5]. (i) *The natural cone \mathfrak{P}^h is pointwise invariant under J .* (ii) *The cone \mathfrak{P}^h is self-dual.* (iii) *The map $\xi \in \mathfrak{P}^h \mapsto \omega_\xi \in \mathfrak{M}_+^+$, the positive part of the predual, is bijective.*

We now state the next Radon-Nikodym theorem for the cone, which is our main result in the paper.

THEOREM 2.3. *The cone \mathfrak{P}^h is precisely the set of all vectors of the form $\xi = tJt\xi_0$, where t is a closed operator affiliated with \mathfrak{M} and satisfies $\xi_0 \in \mathfrak{D}(t)$ and $JtJ\xi_0 = Jt\xi_0 \in \mathfrak{D}(t)$.*

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We now prove that a vector $tJtJ\xi_0$ described in the theorem belongs to the cone, and the rest of the paper will be devoted to the construction of an (unbounded) operator t by starting from an arbitrary vector in the cone. To prove $tJtJ\xi_0 \in \mathfrak{P}^h$, we set $t_n = te_n \in \mathfrak{N}$, $n = 1, 2, \dots$, where e_n is the spectral projection of $|t| = (t^*t)^{1/2}$ corresponding to the interval $[0, n]$. It is easily shown that $t_nJt_nJ\xi_0$ tends to $tJtJ\xi_0$ (as $n \rightarrow \infty$), due to the fact that t and JtJ are affiliated with \mathfrak{M} and \mathfrak{M}' respectively. Since each $t_nJt_nJ\xi_0$ belongs to the cone, the result follows from the closedness of the cone.

By Proposition 2.2 (iii), any $\varphi \in \mathfrak{N}_*^+$ admits a unique implementing vector in \mathfrak{P}^h which we will denote by ξ_φ . (Therefore, $\xi_{\varphi_0} = \xi_0$.) Then the map $x\xi_0 \in \mathfrak{N}\xi_0 \rightarrow x^*\xi_\varphi \in \mathfrak{N}\xi_\varphi$ is a densely-defined closable (conjugate linear) operator. It is easy to show that the phase part of the polar decomposition of the closure is J (by Theorem 1 [1]). The positive selfadjoint part, denoted by $\Delta_{\varphi\varphi_0}^{1/2}$, is known as (the square root of) the relative modular operator (of φ with respect to φ_0). Finally we remark that, when $\varphi = \varphi_0$, $\Delta_{\varphi_0\varphi_0} = \Delta_{\varphi_0}$ is the usual modular operator Δ .

3. Technical lemmas. In this section we collect some technical lemmas. We choose and fix $\varphi \in \mathfrak{N}_*^+$, $\xi_\varphi \in \mathfrak{P}^h$ throughout the section.

LEMMA 3.1. *The map $x \in \mathfrak{N} \mapsto (\Delta^{1/4}x\xi_0|\xi_\varphi)$ gives rise to an element in \mathfrak{N}_*^+ , which we shall denote by ψ .*

PROOF. One can easily show that this linear form is weakly continuous. (See the proof of Lemma 2.10 [5], for example.) Also, the self-duality of \mathfrak{P}^h (Proposition 2.1 (ii)) and Definition 2.1 guarantee the positivity of ψ . Q.E.D.

Thanks to recent theories of noncommutative L^p -spaces, [3, 6, 7, 8, 9], it is known that both of $\Delta_{\varphi\varphi_0}^{1/4}\Delta^{1/4}$ and $\Delta^{1/4}\Delta_{\varphi\varphi_0}^{1/4}$ are densely-defined closable operators on \mathfrak{H} . The relation

$$(1) \quad (\Delta_{\varphi\varphi_0}^{1/4}\Delta^{1/4})^* = (\Delta^{1/4}\Delta_{\varphi\varphi_0}^{1/4})^-$$

is also known. The next result is a special case of the more general result obtained in [7]. However, for the sake of completeness, we present its proof.

LEMMA 3.2. *For each $x \in \mathfrak{N}$, the vector $\Delta_{\varphi\varphi_0}^{1/4}x\xi_0$ belongs to the domain of $\Delta^{1/4}$.*

PROOF. For each $y \in \mathfrak{N}$, we set

$$f(z) = (\Delta_{\varphi\varphi_0}^{z/2}x\xi_0|\Delta^{(1-\bar{z})/2}y\xi_0),$$

and observe that it is bounded continuous on $0 \leq \operatorname{Re} z \leq 1$ and analytic on $0 < \operatorname{Re} z < 1$. For $z = it \in i\mathbb{R}$, we estimate

$$\begin{aligned} |f(it)| &= |(\Delta_{\varphi\varphi_0}^{it/2}x\xi_0|\Delta^{it/2}\Delta^{1/2}y\xi_0)| = |(\Delta_{\varphi\varphi_0}^{it/2}x\xi_0|\Delta^{it/2}Jy^*\xi_0)| \\ &= |(\Delta^{-it/2}\Delta_{\varphi\varphi_0}^{it/2}x\xi_0|Jy^*J\xi_0)| = |(\Delta^{-it/2}\Delta_{\varphi\varphi_0}^{it/2}xJyJ\xi_0|\xi_0)| \quad (JyJ \in \mathfrak{N}') \\ &\leq \|x\|\|\xi_0\|\|y\xi_0\|. \end{aligned}$$

Here we used the fact that $\Delta^{-it/2}\Delta_{\varphi\varphi_0}^{it/2} = (D\varphi: D\varphi_0)_{-t/2}^*$ belongs to \mathfrak{N} . Also for $z = 1 + it \in 1 + i\mathbb{R}$, we estimate

$$|f(1 + it)| = |(\Delta_{\varphi\varphi_0}^{it/2}\Delta_{\varphi\varphi_0}^{1/2}x\xi_0|\Delta_{\varphi\varphi_0}^{it/2}y\xi_0)| = |(\Delta_{\varphi\varphi_0}^{it/2}Jx^*\xi_\varphi|\Delta_{\varphi\varphi_0}^{it/2}y\xi_0)| \leq \|x\|\|\xi_\varphi\|\|y\xi_0\|.$$

It now follows from the Phragmén-Lindelöf theorem that

$$|f(\frac{1}{2})| = |(\Delta_{\varphi\varphi_0}^{1/4}x\xi_0|\Delta_{\varphi\varphi_0}^{1/4}y\xi_0)| \leq \|x\|\|y\xi_0\| \max(\|\xi_0\|, \|\xi_\varphi\|).$$

Since $\mathfrak{N}\xi_0$ is a core for the selfadjoint operator $\Delta^{1/2}$ (hence for $\Delta^{1/4}$), the above inequality implies that $\Delta_{\varphi\varphi_0}^{1/4}x\xi_0$ belongs to $\mathfrak{D}(\Delta^{1/4})$. Q.E.D.

LEMMA 3.3. *The positive selfadjoint part $|(\Delta_{\varphi\varphi_0}^{1/4}\Delta^{1/4})^-|$ of the polar decomposition of $(\Delta_{\varphi\varphi_0}^{1/4}\Delta^{1/4})^-$ is exactly $\Delta_{\psi\varphi_0}^{1/2}$. (See Lemma 3.1 for the definition of ψ .) Furthermore, the vector ξ_ψ belongs to $\mathfrak{D}(\Delta^{1/4}) \cap \mathfrak{D}(\Delta^{-1/4})$.*

PROOF. Let $(\Delta_{\varphi\varphi_0}^{1/4}\Delta^{1/4})^- = uh$ be the polar decomposition. Clearly uh is $(-\frac{1}{2})$ -homogeneous and ξ_0 belongs to $\mathfrak{D}(h)$, that is, h^2 is integrable (see [3]). We thus conclude that $u \in \mathfrak{N}$ and $h = \Delta_{x\varphi_0}^{1/2}$ with a unique $\chi \in \mathfrak{N}_*^+$ (Corollary 18 [3], or §§1, 2 of [8]).

For $x \in \mathfrak{N}_0$, we compute

$$\begin{aligned} (xh\xi_0|h\xi_0) &= (xJh\xi_0|Jh\xi_0) \\ &= (h\xi_0 = \Delta_{x\varphi_0}^{1/2}\xi_0 = \xi_\chi \in \mathfrak{P}^h \text{ and Proposition 2.2 (i)}) \\ &= (xJu^*(\Delta_{\varphi\varphi_0}^{1/4}\Delta^{1/4})^-\xi_0|Ju^*(\Delta_{\varphi\varphi_0}^{1/4}\Delta^{1/4})^-\xi_0) \\ (2) \quad &= (u^*(\Delta_{\varphi\varphi_0}^{1/4}\Delta^{1/4})^-\xi_0|JxJu^*(\Delta_{\varphi\varphi_0}^{1/4}\Delta^{1/4})^-\xi_0) \\ &= (uu^*(\Delta_{\varphi\varphi_0}^{1/4}\Delta^{1/4})^-\xi_0|JxJ(\Delta_{\varphi\varphi_0}^{1/4}\Delta^{1/4})^-\xi_0) \quad (u \in \mathfrak{N}, JxJ \in \mathfrak{N}') \\ &= ((\Delta_{\varphi\varphi_0}^{1/4}\Delta^{1/4})^-\xi_0|JxJ(\Delta_{\varphi\varphi_0}^{1/4}\Delta^{1/4})^-\xi_0) = (\Delta_{\varphi\varphi_0}^{1/4}\xi_0|JxJ\Delta_{\varphi\varphi_0}^{1/4}\xi_0). \end{aligned}$$

The operator $\Delta_{\varphi\varphi_0}^{1/4}$ being $(-\frac{1}{4})$ -homogeneous [3], Lemma 2.1 [8] yields

$$JxJ\Delta_{\varphi\varphi_0}^{1/4}\xi_0 = \Delta_{\varphi\varphi_0}^{1/4}J\sigma_{-i/4}(x)J\xi_0.$$

Therefore, (2) implies

$$\begin{aligned} (xh\xi_0|h\xi_0) &= (\Delta_{\varphi\varphi_0}^{1/4}\xi_0|\Delta_{\varphi\varphi_0}^{1/4}J\sigma_{-i/4}(x)\xi_0) = (\Delta_{\varphi\varphi_0}^{1/2}\xi_0|J\sigma_{-i/4}(x)\xi_0) \\ &= (\xi_\varphi|J\sigma_{-i/4}(x)\xi_0) = (\sigma_{-i/4}(x)\xi_0|\xi_\varphi) \quad (\text{Proposition 2.2 (i)}) \\ &= (\Delta^{1/4}x\xi_0|\xi_\varphi) = \psi(x). \end{aligned}$$

Clearly $(xh\xi_0|h\xi_0) = \psi(x)$ remains valid for all $x \in \mathfrak{N}$ so that $h = \Delta_{x\varphi_0}^{1/2} = \Delta_{\psi\varphi_0}^{1/2}$.

To complete the proof, it suffices to show $\xi_\psi \in \mathfrak{D}(\Delta^{-1/4})$ due to Proposition 2.2 (i) and $J\mathfrak{D}(\Delta^{-1/4}) = \mathfrak{D}(\Delta^{1/4})$. By (1) and the first half of the proof, the right polar decomposition of $(\Delta^{1/4}\Delta_{\varphi\varphi_0}^{1/4})^-$ is

$$(\Delta^{1/4}\Delta_{\varphi\varphi_0}^{1/4})^- = \Delta_{\psi\varphi_0}^{1/2}u^*.$$

We thus have

$$\xi_\psi = \Delta_{\psi\varphi_0}^{1/2}\xi_0 = (\Delta^{1/4}\Delta_{\varphi\varphi_0}^{1/4})^-u\xi_0.$$

It follows from Lemma 3.2 that we actually have $\xi_\psi = \Delta^{1/4} \Delta_{\psi\varphi_0}^{1/4} u \xi_0$ without the closure sign so that

$$\xi_\psi \in \mathfrak{R}(\Delta^{1/4}) = \mathfrak{O}(\Delta^{-1/4}). \quad \text{Q.E.D.}$$

COROLLARY 3.4. *The set $\mathfrak{N}'_0 \xi_0$ is included in $\mathfrak{O}(\Delta^{-1/4} \Delta_{\psi\varphi_0}^{1/2} \Delta^{-1/4})$.*

PROOF. Lemma 3.3 shows that $\xi_0 \in \mathfrak{O}(\Delta^{-1/4} \Delta_{\psi\varphi_0}^{1/2} \Delta^{-1/4})$ so that the corollary follows from a homogeneity argument. (See Lemma 2.1 [8].) Q.E.D.

4. Proof of Theorem 2.3. This section is devoted to the proof of the nontrivial half of Theorem 2.3. We choose and fix an arbitrary vector ξ in the cone \mathcal{P}^h . Let $\varphi = \omega_\xi \in \mathfrak{N}_*^+$ so that we will write $\xi = \xi_\varphi$ henceforth and try to construct an operator t satisfying $\xi_\varphi = tJ\xi_0$. Easy computations for the matrix algebra suggest that the closure (or the adjoint) of the densely-defined (Corollary 3.4) symmetric operator $\Delta^{-1/4} \Delta_{\psi\varphi_0}^{1/2} \Delta^{-1/4}$ is an obvious candidate for t . However, it does not seem to work, the reason being lack of information on a core for the operator $\Delta^{-1/4} \Delta_{\psi\varphi_0}^{1/2} \Delta^{-1/4}$. (See the very last part of §4.) Instead, we define

$$t = ((\Delta^{-1/4} \Delta_{\psi\varphi_0}^{1/2} \Delta^{-1/4})|_{\mathfrak{N}'_0 \xi_0})^*,$$

where ψ is constructed from φ as in Lemma 3.1. We then have

$$(3) \quad \Delta^{-1/4} \Delta_{\psi\varphi_0}^{1/2} \Delta^{-1/4} \subseteq (\Delta^{-1/4} \Delta_{\psi\varphi_0}^{1/2} \Delta^{-1/4})^* \subseteq ((\Delta^{-1/4} \Delta_{\psi\varphi_0}^{1/2} \Delta^{-1/4})|_{\mathfrak{N}'_0 \xi_0})^* = t.$$

In particular, $\mathfrak{N}'_0 \xi_0 \subseteq \mathfrak{O}(t)$ and it is easy to see

$$(4) \quad x' \Delta^{-1/4} \Delta_{\psi\varphi_0}^{1/2} \Delta^{-1/4} y' \xi_0 = \Delta^{-1/4} \Delta_{\psi\varphi_0}^{1/2} \Delta^{-1/4} x' y' \xi_0, \quad x', y' \in \mathfrak{N}'_0,$$

by making use of homogeneity.

PROOF OF THEOREM 2.3. For $\zeta \in \mathfrak{O}(t)$ and $x', y' \in \mathfrak{N}'_0$, (3) and (4) together imply

$$\begin{aligned} (x' \zeta | \Delta^{-1/4} \Delta_{\psi\varphi_0}^{1/2} \Delta^{-1/4} y' \xi_0) &= (\zeta | \Delta^{-1/4} \Delta_{\psi\varphi_0}^{1/2} \Delta^{-1/4} x' y' \xi_0) \\ &= (t \zeta | x' y' \xi_0) \quad (\text{the definition of } t) \\ &= (x' t \zeta | y' \xi_0), \end{aligned}$$

so that $x' \zeta \in \mathfrak{O}(t)$ and $tx' \zeta = x' t \zeta$. Thus, the closedness of t and density of \mathfrak{N}'_0 in \mathfrak{N}' (with respect to the strong operator topology) imply that t is affiliated with \mathfrak{N} .

By (3) we have

$$(5) \quad t \xi_0 = \Delta^{-1/4} \xi_\psi, \quad JtJ\xi_0 = J\Delta^{-1/4} \xi_\psi = \Delta^{1/4} \xi_\psi.$$

(See Lemma 3.3.) For each $x' \in \mathfrak{N}'_0$, we then compute

$$\begin{aligned} (\Delta^{1/4} \xi_\psi | \Delta^{-1/4} \Delta_{\psi\varphi_0}^{1/2} \Delta^{-1/4} x' \xi_0) &= (\xi_\psi | \Delta_{\psi\varphi_0}^{1/2} \Delta^{-1/4} x' \xi_0) = (\xi_\psi | \Delta_{\psi\varphi_0}^{1/2} \Delta^{-1/4} \Delta^{1/2} Jx'^* \xi_0) \\ &= (\xi_\psi | \Delta_{\psi\varphi_0}^{1/2} \Delta^{1/4} Jx'^* J \xi_0) = (\xi_\psi | \Delta_{\psi\varphi_0}^{1/2} \sigma_{-i/4}(Jx'^* J) \xi_0) \\ &= (\xi_\psi | J \sigma_{-i/4}(Jx'^* J)^* \xi_\psi) = (\xi_\psi | J \sigma_{i/4}(Jx' J) \xi_\psi) \\ &= (\sigma_{i/4}(Jx' J) \xi_\psi | \xi_\psi) \quad (\text{Proposition 2.2 (i)}) \\ &= \psi(\sigma_{i/4}(Jx' J)) = (\Delta^{1/4} \sigma_{i/4}(Jx' J) \xi_0 | \xi_\varphi) \quad (\text{the definition of } \psi) \\ &= (Jx' J \xi_0 | \xi_\varphi) = (\xi_\varphi | x' \xi_0) \quad (\text{Proposition 2.2 (i)}). \end{aligned}$$

Thus, the definition of t shows that $\Delta^{1/4}\xi_\psi \in \mathcal{D}(t)$ and $t\Delta^{1/4}\xi_\psi = \xi_\psi$. These facts together with (5) complete the proof. Q.E.D.

5. Concluding remarks. Provided that \mathfrak{M} is a factor of type III_λ , $0 < \lambda < 1$, for a cyclic (and separating) vector ξ_ψ in \mathfrak{P}^h , there always exists an (invertible) bounded operator t in \mathfrak{M} so that $\xi_\psi = tJtJ\xi_0$. In fact, due to a result of Connes-Takesaki (see Theorem 3.3 [6]), there exists a unitary operator u in \mathfrak{M} satisfying $\varphi_0 \leq l_1 u\varphi u^*$ and $u\varphi u^* \leq l_2 \varphi_0$ with some $l_1, l_2 \geq 0$. It is easy to check (Lemma 3.12 [2])

$$(D(u\varphi u^*): D\varphi_0)_{-i/4} J(D(u\varphi u^*): D\varphi_0)_{-i/4} J\xi_0 = uJtJ\xi_\psi$$

so that $t = u^*(D(u\varphi u^*): D\varphi_0)_{-i/4}$ does the job.

We also know the necessary and sufficient condition for a generic ξ in \mathfrak{P}^h to admit a positive bounded t in \mathfrak{M} satisfying $\xi = tJtJ\xi_0$ (Proposition 3.1 [8]). However, our construction of the operator t in §4 does not guarantee the positive selfadjointness of t (unless Δ is bounded). The problem of finding a positive selfadjoint t seems to deserve further investigation.

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