PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 84, Number 2, February 1982

A NONVARIATIONAL SECOND ORDER ELLIPTIC OPERATOR WITH SINGULAR ELLIPTIC MEASURE

LUCIANO MODICA, STEFANO MORTOLA AND SANDRO SALSA

ABSTRACT. We exhibit an example which proves that the elliptic measure for a second-order operator of the form $\sum_{i,j=1}^{n} a_{ij} D_{ij}^2$ with continuous coefficients can be singular with respect to the surface measure on the boundary of a smooth two-dimensional domain.

1. Introduction. In the papers [2] and [4], examples of singular elliptic measures for second-order operators in divergence form are given. In this note, following the ideas contained in [4], we exhibit an analogous example in the nonvariational case.

We recall the definition of elliptic measure. Let Ω be a bounded subset of R^n with smooth boundary $\partial\Omega$ and $L = \sum_{i,j=1}^{n} a_{ij} D_{ij}^2$, a uniformly elliptic operator with continuous coefficients in $\overline{\Omega}$. It is well known that, for every $g \in C(\partial\Omega)$, there exists a unique solution $u \in W_{\mathrm{loc}}^{2,2}(\Omega) \cap C(\overline{\Omega})$ of the problem

$$Lu = 0 \quad \text{in } \Omega, \qquad u = g \quad \text{on } \partial\Omega.$$

The classical maximum principle and the Riesz representation theorem imply that for each $P \in \Omega$ there exists a Borel measure on $\partial\Omega$, ω_L^P (the L-elliptic measure evaluated at P), such that the following formula holds:

$$u(P) = \int_{\partial\Omega} g(\sigma)\, d\omega_L^P(\sigma).$$

On the other hand, by a result of Pucci and Alexandrov (see [1] and [6]), the solution, vanishing on the boundary, of the equation $Lv = f$, $f \in L^n(\Omega)$, satisfies the following estimate

$$\max_{P \in \overline{\Omega}} |v(P)| \leq c\|f\|_{L^n(\Omega)} \tag{1.1}$$

where the constant c depends only on the ellipticity constants and the geometry of Ω.

Notice that, by a result of Talenti [7], in dimension two we have the stronger inequality

$$\|v\|_{W^{2,2}(\Omega)} \leq c\|f\|_{L^2(\Omega)}, \tag{1.2}$$

where c depends on the same parameters as before.

Pucci-Alexandrov's theorem implies the existence of the Green's function $G(P; Q)$, such that $G(P; \cdot) \in L^{n/(n-1)}(\Omega)$ for every fixed P and $v(P) = \int_\Omega G(P; Q) f(Q)\, dQ$.

Received by the editors January 12, 1981.
1980 *Mathematics Subject Classification*. Primary 35J25.

In the case of smooth coefficients, the divergence theorem gives the connection between the Green's function and the L-elliptic measure; we have

$$d\omega_L^P(\sigma) = \sum_{i,j=1}^{n} a_{ij}(\sigma)\nu_i D_{\sigma_j} G(P; \sigma)\, d\sigma$$

where $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$ is the inward unit normal to $\partial\Omega$, and $d\sigma$ is the usual $(n-1)$-dimensional surface measure on $\partial\Omega$.

Furthermore we recall that in $\Omega \setminus \{P\}$, $G(P, \cdot)$ satisfies the equation $L^*G(P; \cdot) = 0$, where L^* is the adjoint operator of L; formally $L^*u = \sum_{i,j=1}^n D_{ij}^2(a_{ij}u)$.

2. Main lemmas. In this section, B is a bounded C^∞ domain in the upper half-plane R_+^2 adjacent to the x-axis such that

$$B_0 = \{(x,y): |x| < 2, y = 0\} \subseteq \partial B.$$

P is a given point in B.

LEMMA 1. *Suppose $\beta = \beta(x,y)$, $\beta^h = \beta^h(x,y)$ $(h = 1, 2, \ldots)$ are C^∞ functions in R^2 satisfying the following conditions*:

(a) $\frac{1}{2} \leq \beta \leq \frac{3}{2}$, $\frac{1}{2} \leq \beta^h \leq \frac{3}{2}$ $\forall h$;

(b) $|D_y\beta^h(x,y)| \leq c_1$, $|D_{yy}^2\beta^h(x,y)| \leq c_2$ $\forall (x,y) \in B, \forall h$;

(c) β^h *converges weakly in* $L^2(B)$ *to* β, *as* $h \to \infty$;

(d) $\beta^h(x,y) = \beta^h(x, -y)$.

Denote by E and E_h respectively the operators $D_{xx}^2 + \beta D_{yy}^2$ and $D_{xx}^2 + \beta^h D_{yy}^2$.

Claim. If $G(x,y) = G(P; x,y)$ and $G^h(x,y) = G^h(P; x,y)$ are the Green's functions for E and E_h in B with pole P, then

$$D_y G^h(x, 0) \to D_y G(x, 0) \quad \textit{uniformly for } x \in [-1, 1].$$

PROOF. Denote by T_+ the set $\{(x,y): |x| < 1 + \varepsilon;\ 0 < y < \mu\}$ with ε, μ small enough to ensure $\overline{T}_+ \subseteq \overline{B}$, $P \notin \overline{T}_+$. T_- will be the set $\{(x,y): |x| < 1 + \varepsilon, -\mu < y < 0\}$ and $T = \overline{T_+ \cup T_-}$. In T_+ we have

$$E_h^* G^h = D_{xx}^2 G^h + D_{yy}^2(\beta^h G^h) = 0. \tag{2.1}$$

Furthermore $G^h(x, 0) = 0$ if $|x| < 2$.

The equation (2.1) can be written in the following divergence form:

$$D_{xx}^2 G^h + D_y(\beta^h D_y G^h) = -D_y(D_y\beta^h G^h). \tag{2.2}$$

Extend now $G^h(P; x,y)$ to an odd function with respect to y across $y = 0$ and call $\tilde{G}^h(P; x,y)$ the extended function; then $\tilde{G}^h(P; x,y)$ satisfies in T the equation (2.2). On the other hand, by (1.1), it is easy to show that $\tilde{G}^h(P; x,y)$ is equibounded in $L^2(T)$ and, by the hypothesis (b) we have $\|D_y\beta^h\|_{L^\infty(T)} \leq c_1$. Well-known results on divergence form equations imply that $\|\tilde{G}^h(P; \cdot)\|_{W^{1,2}(T)} < c$ (independent of h) and thus, by Sobolev's immersion theorem, $\tilde{G}^h(P; \cdot)$ is equibounded in $L^p(T)$ for every $p > 2$.

Therefore, Meyers' theorem (see [3]) implies the existence of $\delta > 0$ such that

$$\|\tilde{G}^h(P; \cdot)\|_{W^{1,2+\delta}(T)} \leq \text{const} \quad (\text{independent of } h).$$

In particular we have

$$\|G^h(P;\ \cdot)\|_{W^{1,2+\delta}(T_+)} \leq \text{const} \quad (\text{independent of } h). \tag{2.3}$$

Differentiating now (2.1) with respect to y and putting $v^h = D_y G^h$, we see that v^h satisfies in T_+ the following divergence form equation:

$$D^2_{xx} v^h + D_y\big(\beta^h D_y v^h\big) = -2D_y\big(D_y \beta^h v^h\big) - D_y\big(D^2_{yy}\beta^h G^h\big).$$

By (2.3) and hypothesis (b), the right-hand side is the divergence of an equibounded (in $L^{2+\delta}(T_+)$) vector field. Using once more Meyers' result we deduce that

$$\|v^h\|_{W^{1,2+\delta}(T_+)} \leq \text{const} \quad (\text{independent of } h). \tag{2.4}$$

From (2.1) and (2.4) it follows that $\|G^h(P;\ \cdot)\|_{W^{2,2+\delta}(T_+)} \leq$ const (independent of h). Sobolev's immersion theorem implies now that G^h (actually a subsequence) converges in $C^1(\overline{T}_+)$ to some continuous function $g(x, y)$. The conclusion of Lemma 1 will follow from the following lemma.

LEMMA 2. *Under the hypothesis of Lemma* 1, $G^h(P; x, y)$ *converges weakly in* $L^2(B)$ *to* $G(P; x, y)$.

PROOF. Consider the function

$$u^h(P) = \int_B G^h(P; x, y) f(x, y)\, dx\, dy \tag{2.5}$$

where $f \in L^2(B)$.

The function $u^h(P)$ is the solution, vanishing on ∂B, of $E_h u = f$ in B. From the result of Talenti [7], we have $\|u^h\|_{W^{2,2}(B)} \leq c\|f\|_{L^2(B)}$ with c depending only on the geometry of B. Therefore u^h admits a subsequence converging weakly in $W^{2,2}(B)$ and strongly in $W^{1,p}(B)$ for every $p > 2$ to a function $u \in W^{2,2}(B)$, which vanishes on ∂B. We will show that u is the solution, vanishing on ∂B, of $Eu = f$.

We have $D^2_{xx} u^h + \beta^h D^2_{yy} u^h = f$, and therefore

$$D^2_{xx} u^h + D_y\big(\beta^h D_y u^h\big) = D_y \beta^h D_y u^h + f.$$

It is enough to show that $(D_y \beta^h)(D_y u^h)$ converges weakly in $L^2(B)$ to $(D_y \beta)(D_y u)$ and this is an easy consequence of the following facts: $\beta^h \to \beta$ in $L^2(B)$ weakly, $\|\beta^h\|_{L^\infty(B)} <$ const, $D_y u^h \to D_y u$ in $L^2(B)$ strongly.

On the other hand $G^h(P;\ \cdot)$ is equibounded in $L^2(B)$ and so has a subsequence which converges weakly in $L^2(B)$ to some function $v_P \in L^2(B)$. From the representation formula (2.5), letting h_n tend to infinity, we have

$$u(P) = \int_B v_P(x, y) f(x, y)\, dx\, dy. \tag{2.6}$$

Since (2.6) holds for any $f \in L^2(B)$, we conclude $v_P = G$.

LEMMA 3. *Suppose* β *and* β^h *are* C^∞ *functions in* R^2 *satisfying the following hypotheses*:

(a) $\frac{1}{2} < \beta < \frac{3}{2}$, $\frac{1}{2} < \beta^h < \frac{3}{2}$ $\forall h$;

(b) $|D_x \beta^h(x, y)| < C_1$, $|D^2_{xx}\beta^h(x, y)| < C_2$ $\forall (x, y) \in B$, $\forall h$;

(c) β^h *converges to* β *in* $L^2_{\mathrm{loc}}(B)$;

(d) $\beta^h(x, y) = \beta^h(x, -y)$.

Denote by E *and* E_h *the operators* $D^2_{xx} + \beta \cdot D^2_{yy}$ *and* $D^2_{xx} + \beta^h \cdot D^2_{yy}$ *and by* $G(x, y) = G(P; x, y)$, $G^h(x, y) = G^h(P; x, y)$ *their Green's functions in* B *with pole* P.

Claim. $D_y(\beta^h G^h)(x, 0) \to D_y(\beta G)(x, 0)$ *uniformly for* $x \in [-1, 1]$.

PROOF. Let T_+ be as in the proof of Lemma 1. In T_+ we have

$$D^2_{xx} G^h + D^2_{yy}(\beta^h G^h) = 0.$$

This means that the function $v^h = \beta^h G^h$ satisfies the equation $D^2_{xx}(v^h/\beta^h) + D^2_{yy} v^h = 0$. Furthermore $v^h(x, y) = 0$ if $|x| < 1 + \varepsilon$ and $y = 0$. Arguing now as in Lemma 1, having interchanged the roles of x and y, we deduce that $v^h \to v$ in $C^1(\overline{T}_+)$ where v is some C^1 function in $\overline{T}_+$.

The lemma will be proved if we show that $D_y v(x, 0) = D_y(\beta G)(x, 0)$ for $|x| < 1$. But, if we recall that the L-elliptic measure ω^P_L has density $D_y(\beta G)(x, 0)$, this follows from the following general lemma.

LEMMA 4. *Let* Ω *be a bounded domain of* R^n *and* $L^h = \sum^n_{i,j=1} a^h_{ij} D^2_{ij}$ $(h = 1, 2, \ldots)$ *with* $a_{ij} \in C(\overline{\Omega})$. *Suppose furthermore that*

(a) *for every* $\xi \in R^n$, $\mu|\xi|^2 \leqslant \sum^n_{i,j=1} a^h_{ij}\xi_i\xi_j < M|\xi|^2$ *with* μ *and* M *independent of* h;

(b) $a^h_{ij} \to a_{ij}$ *in* $L^2_{\mathrm{loc}}(\Omega)$.

Then, if L *is the limit operator* $\sum^n_{i,j=1} a_{ij} D^2_{ij}$ *and* P *is fixed in* Ω, $\omega^P_{L^h}$ *converges weakly to* ω^P_L, *that is, for every* $g \in C(\partial\Omega)$ *we have*

$$\int_\Omega g(\sigma)\, d\omega^P_{L^h} \to \int_\Omega g(\sigma)\, d\omega^P_L.$$

PROOF. It is enough to show that, for every $\varphi \in C^\infty(\partial\Omega)$, the solutions of the problems $L^h u^h = 0$ in Ω, $u^h = \varphi$ on $\partial\Omega$ $(h = 1, 2, \ldots)$ converge in P to the solution u of the problem $Lu = 0$ in Ω, $u = \varphi$ on $\partial\Omega$.

Since φ is a smooth function, u_h is an equibounded sequence in $W^{2,p}(\Omega)$ for every $p < \infty$ and therefore converges weakly in $W^{2,p}(\Omega)$ and strongly in $L^\infty(\Omega)$ to some function $u \in W^{2,p}(\Omega)$. Obviously u is the solution of the problem $Lu = 0$ in Ω, $u = \varphi$ on $\partial\Omega$, and the proof is complete.

3. Construction of a singular elliptic measure. Suppose $\{h_n\}$ and $\{k_n\}$ are two increasing sequences of positive integers with $h_n \to \infty$ and $k_n \to \infty$. Let $\varphi_n(x) = 1 + (1/2n^{1/2})\cos(h_n x)$ and denote by ψ a $C^\infty_0(R)$ function such that $\psi(t) = \psi(-t)$, $0 \leqslant \psi \leqslant 1$, $\psi = 1$ if $|t| \leqslant 1$, $\psi = 0$ if $|t| \geqslant 2$. Consider now the function $\alpha(x, y)$ defined by

$$\alpha(x, y) = \begin{cases} \varphi_1(x) & \text{if } |y| > 1/k_1, \\ \psi(k_{n+1}y)\varphi_{n+1}(x) + [1 - \psi(k_{n+1}y)]\varphi_n(x) & \text{if } 1/k_{n+1} \leqslant |y| < 1/k_n, \\ & n = 1, 2, \ldots, \\ 1 & \text{if } y = 0. \end{cases}$$

If $k_{n+1} \geqslant 2k_n$, the function α is continuous in R^2 and C^∞ except on the x-axis; moreover $\frac{1}{2} \leqslant \alpha \leqslant \frac{3}{2}$.

Let B be as in §2 and $L = D^2_{xx} + \alpha D^2_{yy}$.

THEOREM. *If $\{h_n\}$ and $\{k_n\}$ are suitably chosen, the L-elliptic measure ω^P_L on ∂B (evaluated at the point $P \in B$) is not absolutely continuous with respect to the Lebesgue measure on* $[-1, 1]$.

REMARK. Since the L-elliptic measures evaluated at different points of B are mutually absolutely continuous (by maximum principle), the choice of the point P is irrelevant.

PROOF OF THE THEOREM. We prove the theorem via an approximation argument. Observe that α is the uniform (in R^2) limit of the sequence of the C^∞ functions defined by

$$\alpha_n(x, y) = \begin{cases} \alpha(x, y) & \text{if } |y| \geqslant 1/k_n, \\ \varphi_n(x) & \text{if } |y| < 1/k_n. \end{cases}$$

As usual we denote by L^n the operator $D^2_{xx} + \alpha_n D^2_{yy}$ and by $\omega^P_{L^n}$ its elliptic measure on ∂B evaluated at P.

Note that on B_0 (i.e. $\bar{B} \cap \{y = 0\}$) the density of $\omega^P_{L^n}$ is given by $\varphi_n(x) D_y G^n(P; x, 0)$, where $G^n(P; x, y)$ denotes the Green's function of L^n.

Applying Lemma 4, we see that $\omega^P_{L^n}$ converges weakly to ω^P_L; therefore the theorem will be proved if we choose $\{h_n\}$ and $\{k_n\}$ such that $\varphi_n(x) D_y G^n(P; x, 0)$ converges weakly to a singular measure on $[-1, 1]$.

We proceed by induction.

Set $h_1 = k_1 = 1$ and suppose we have already chosen $h_2, \ldots, h_n$; $k_2, \ldots, k_n$ in such a way that $h_j > 4h_{j-1}$, $k_j > 2k_{j-1}$ for $j = 1, 2, \ldots, n$. To choose k_{n+1}, put $c = \min_{x \in [-1,1]} D_y G^1(x, 0)$, which is a positive number by Hopf's lemma (see [5, p. 65]) and define

$$\bar{\alpha}_k(x, y) = \begin{cases} \alpha_n(x, y) & \text{if } |y| \geqslant 1/k_n, \\ \psi(ky) + [1 - \psi(ky)]\varphi_n(x) & \text{if } |y| < 1/k_n. \end{cases}$$

If $k \geqslant 2k_n$, it is easy to check that $\bar{\alpha}_k \in C^\infty(R^2)$ and moreover $\bar{\alpha}_k$ converges to α_n in $L^2_{\text{loc}}(R^2)$ as $k \to \infty$ and $D_x \bar{\alpha}_k$, $D^2_{xx} \bar{\alpha}_k$ are equibounded in R^2. Let us denote by $\bar{G}^k(P; x, y)$ the Green's function in B for the operator $D^2_{xx} + \bar{\alpha}_k D^2_{yy}$, with pole P. Lemma 3 guarantees now the existence of an index k_{n+1} such that $k_{n+1} > 2k_n$ and

$$\max_{x \in [-1,1]} \left| D_y\left(\bar{\alpha}_{k_{n+1}} \bar{G}^{k_{n+1}}\right)(x, 0) - D_y(\alpha_n G^n)(x, 0) \right| \leqslant \frac{c}{4^{n+2}}.$$

That is, since $\bar{\alpha}_{k_{n+1}} = 1$ near $\{y = 0\}$ and $\bar{G}^{k_{n+1}} = G^n = 0$ on $\{y = 0\}$,

$$\max_{x \in [-1,1]} \left| D_y G^{k_{n+1}}(P; x, 0) - \varphi_n(x) D_y G^n(P; x, 0) \right| \leqslant \frac{c}{4^{n+2}}. \tag{3.1}$$

To choose h_{n+1} we define

$$\tilde{\alpha}_h(x, y) = \begin{cases} \alpha_n(x, y) \quad \text{if } |y| \geqslant 1/k_n, \\ \psi(k_{n+1} y)\left[1 + \dfrac{1}{2(n+1)^{1/2}} \cos(hx)\right] + [1 - \psi(k_{n+1} y)]\varphi_n(x) \\ \hfill \text{if } |y| < 1/k_n. \end{cases}$$

Clearly $\tilde{\alpha}_h \in C^\infty(R^2)$ and $\tilde{\alpha}_h \to \bar{\alpha}_{k_{n+1}}$ weakly in $L^2(B)$ as $h \to \infty$; furthermore, $D_y \tilde{\alpha}_h$ and $D_{yy}^2 \tilde{\alpha}_h$ are equibounded in B. Therefore, Lemma 1 implies the existence of an index h_{n+1} such that $h_{n+1} > 4h_n$ and

$$\max_{x \in [-1,1]} \left| D_y \tilde{G}^{h_{n+1}}(P; x, 0) - D_y \bar{G}^{k_{n+1}}(P; x, 0) \right| < \frac{c}{4^{n+2}}, \tag{3.2}$$

where $\tilde{G}^h(P; x, 0)$ is the Green's function in B for the operator $D_{xx}^2 + \tilde{\alpha}_h D_{yy}^2$ with pole P.

By this choice of h_{n+1} we have $\tilde{\alpha}_{h_{n+1}} = \alpha_{n+1}$; (3.1) and (3.2) give

$$\max_{x \in [-1,1]} \left| D_y G^{n+1}(P; x, 0) - D_y G^n(P; x, 0)\varphi_n(x) \right| < \frac{c}{4^{n+1}}. \tag{3.3}$$

From (3.3) we deduce that

$$\begin{aligned} &\varphi_{n+1} D_y G^{n+1}(P; x, 0) \\ &\quad = \left(\prod_{j=1}^n \varphi_j(x) \right) \left[D_y G^1(P; x, 0) + \sum_{j=1}^n R_j(x) \left(\prod_{h=1}^j \varphi_h(x) \right)^{-1} \right] \end{aligned} \tag{3.4}$$

where $\max_{x \in [-1,1]} |R_j(x)| < c/4^{j+1}$.

Since $\prod_{h=1}^j \varphi_h > 2^{-j}$ it follows that the function between square brackets in (3.4) converges uniformly in $[-1, 1]$ to some continuous function $w(x)$; moreover $w(x) > \frac{3}{4}c > 0$. On the other hand, $\prod_{j=1}^n \varphi_j$ converges (see [**8**, vol. I, p. 209]) weakly in the sense of the measures on $[-1, 1]$ to a singular measure.

We conclude that ω_L^P also converges weakly to a singular measure on $[-1, 1]$ and so the proof is complete.

REFERENCES

1. A. D. Alexandrov, *Majorization of solutions of second-order linear equations*, Amer. Math. Soc. Transl. (2) **68** (1968), 120–143.

2. L. Caffarelli, E. Fabes and C. Kenig, *Completely singular elliptic harmonic measures*, Indiana Univ. Math. J. (to appear).

3. N. G. Meyers, *An L^p-estimate for the gradient of solutions of second order elliptic divergence equations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. Ser. III **17** (1963), 189–206.

4. L. Modica and S. Mortola, *Construction of a singular elliptic-harmonic measure*, Manuscripta Math. **33** (1980), 81–98.

5. M. Protter and H. F. Weinberger, *Maximum principles in differential equations*, Prentice-Hall, Englewood Cliffs, N.J., 1967.

6. C. Pucci, *Limitazioni per soluzioni di equazioni ellittiche*, Ann. Mat. Pura Appl. (4) **74** (1966), 15–30.

7. G. Talenti, *Equazioni lineari ellittiche in due variabili*, Le Matematiche **21** (1966), 339–376.

8. A. Zygmund, *Trigonometric series*, Cambridge Univ. Press, London and New York, 1959.

ISTITUTO DI MATEMATICA "LEONIDA TONELLI", UNIVERSITÀ DI PISA, I-56100 PISA, ITALY (Current address of Luciano Modica and Stefano Mortola)

ISTITUTO DI MATEMATICA "FEDERICO ENRIQUES", UNIVERSITÀ DI MILANO, I-20133 MILANO, ITALY (Current address of Sandro Salsa)