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A CHARACTERIZATION OF THE UNIFORM CLOSURE OF THE SET OF HOMEOMORPHISMS OF A COMPACT TOTALLY DISCONNECTED METRIC SPACE INTO ITSELF

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ABSTRACT. The limit index $\lambda(x)$ of a point x in a compact metric space is defined. (Roughly: Isolated points have index 0, limit points have index 1, limit points of limit points have index 2, and so forth.) Then the following theorem is proved.

THEOREM 1. *Let E be a compact, totally disconnected metric space. Then the uniform closure of the set of homeomorphisms of E into itself is the set C_λ of continuous functions f from E to E satisfying*

(1) $\lambda(x) \leq \lambda(f(x))$ *for all* $x \in E$, *and*

(2) *if y is not a condensation point of E, then $f^{-1}(y)$ contains at most one x such that* $\lambda(x) = \lambda(y)$.

Further, the set of homeomorphisms of E into E is a dense G_δ subset of the complete metric space C_λ.

A concept that we will call the limit index of a point in a compact metric space was used by Miles in the proof of a theorem in abstract harmonic analysis [**1**, Theorem A]. Theorem 1 of this paper can be proved from that theorem. The proof of Theorem 1 presented in this paper is simpler but similar and does not use harmonic analysis. The original form of the category argument used here is due to Kaufman [**2**]. Adaptations have appeared in [**1**, **3** and **4**].

We first introduce some definitions and notation.

Let E be a compact metric space. For each ordinal $\alpha \leqslant \Omega$ (the first uncountable ordinal), define E_α as follows. Let $E_0 = E$. Let $E_{\alpha+1}$ be the set of limit points of E_α. If β is a limit ordinal, let $E_\beta = \bigcap_{\alpha<\beta} E_\alpha$. (These definitions are due originally to Cantor [**5**]. See also Kuratowski [**6**, p. 261].)

It is shown in [**1**] and in [**6**, p. 262] that $E_\alpha = E_{\alpha+1}$ for some $\alpha < \Omega$. Let α_E be the first ordinal for which this holds and write $\tilde{E}$ for E_{α_E}. Observe that $\tilde{E}$ is the set of condensation points of E.

For a nonempty closed subset F of E, define the limit index of F, denoted $\lambda(F)$, as follows: If $F \cap \tilde{E} \neq \varnothing$, let $\lambda(F) = \alpha_E$; otherwise let $\lambda(F)$ be the last α such that $F \cap E_\alpha \neq \varnothing$. (A compactness argument, given in [**1**], shows that such an α exists.) For $x \in E$, we write $\lambda(x)$ for $\lambda(\{x\})$.

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Observe that λ has the following properties:

(i) If $\alpha \leqslant \alpha_E$, then $\lambda(x) \geqslant \alpha$ if and only if $x \in E_\alpha$.

(ii) $\lambda(F) < \alpha_E$ implies that $F \cap E_{\lambda(F)}$ is finite.

(iii) $y \in F$ implies that $\lambda(y) \leqslant \lambda(F)$.

Let $C(E, E)$ be the set of continuous functions from E to E and $C(E, R)$ be the set of continuous real-valued functions on E. Let C_{fin} be the set of continuous real-valued functions on E with finite range. For $h \in C(E, R)$ and $\varepsilon > 0$, let $G(h, \varepsilon) = \{f \in C_\lambda : \|\gamma \circ f - h\|_\infty < \varepsilon \text{ for some } \gamma \in C_{\text{fin}}\}$.

Let d be a metric on E compatible with the topology of E. For f and g in $C(E, E)$, let $D(f, g) = \sup\{d(f(x), g(x)) : x \in E\}$.

THEOREM 2. *Every homeomorphism of E into itself is an element of C_λ.*

PROOF. Let f be a homeomorphism of E into E. The second condition in the definition of C_λ is trivially satisfied, since f is one-to-one. It remains to show that the first condition holds or, equivalently, that $f(E_\alpha) \subset E_\alpha$ for all α. Assume that $f(E_\alpha) \subset E_\alpha$ is false for some α and let β be the first ordinal for which this happens. We will show that this leads to a contradiction. We have $f(E_\beta) \not\subset E_\beta$, but, for $\alpha < \beta$, $f(E_\alpha) \subset E_\alpha$. Thus, there is an $x \in E_\beta$ such that $y = f(x) \notin E_\beta$. Let $\lambda(y) = \alpha$. Then $\alpha < \beta$. Consider $g = f|_{E_\alpha}$. Clearly, g is a homeomorphism of E_α into E_α. Since y is an isolated point of E_α, $g^{-1}(y) = x$ is an isolated point of E_α. But $x \in E_\beta$ and is therefore a limit point of E_α, so we have a contradiction.

THEOREM 3. *C_λ is complete in the topology of uniform convergence.*

PROOF. See [1].

LEMMA 1. *Let $x_1, \ldots, x_n$ be distinct elements of E; let $g \in C_\lambda$ and let $\eta > 0$. Then there are distinct elements $y_1, \ldots, y_n$ of E such that $\lambda(x_j) \leqslant \lambda(y_j)$ and $d(y_j, g(x_j)) < \eta$ for $1 \leqslant j \leqslant n$.*

PROOF. See [1].

LEMMA 2. *Each $G(h, \varepsilon)$ is dense in C_λ.*

PROOF. Fix $h \in C(E, R)$ and $\varepsilon > 0$. Let $g \in C_\lambda$ and $\eta > 0$. We will show that there is an $f \in G(h, \varepsilon)$ such that $D(f, g) < \eta$.

Write $E = \bigcup_{j=1}^n F_j$, where the F_j are pairwise disjoint, nonvoid, open and closed subsets of E, and where h varies less than ε and g varies less than $\eta/2$ on each F_j. Let $\lambda(F_j) = \alpha_j$. If $\alpha_j < \alpha_E$, then $F_j \cap E_{\alpha_j}$ is finite, so that we may suppose without loss of generality that $F_j \cap E_{\alpha_j}$ consists of a single point x_j. If $\alpha_j = \alpha_E$, let x_j be any point of $F_j \cap E_{\alpha_j}$. By Lemma 1, there are distinct $y_1, \ldots, y_n$ such that $\lambda(y_j) \geqslant \lambda(x_j)$ and $d(y_j, g(x_j)) < \eta/2$, $1 \leqslant j \leqslant n$. Define $f(x) = y_j$ when $x \in F_j$. Then $f \in C_\lambda$ and $D(f, g) < \eta$. Now write $E = \bigcup_{j=1}^n A_j$, where the A_j are disjoint open and closed sets and $y_j \in A_j$, $1 \leqslant j \leqslant n$. Define $\gamma \in C_{\text{fin}}$ by $\gamma(y) = h(x_j)$ when $y \in A_j$. Then, when $x \in F_j$, we have $|\gamma \circ f(x) - h(x)| = |h(x_j) - h(x)| < \varepsilon$, so $\|\gamma \circ f - h\|_\infty < \varepsilon$.

LEMMA 3. *Each $G(h, \varepsilon)$ is open in C_λ.*

PROOF. Fix $h \in C(E, R)$ and $\varepsilon > 0$. Let $g \in G(h, \varepsilon)$ and let $\gamma \in C_{\text{fin}}$ be such that $\|\gamma \circ g - h\|_\infty < \varepsilon$. Let the range of γ be $\{y_1, \ldots, y_n\}$ and let $F_j = \gamma^{-1}(y_j)$, $1 \leqslant j \leqslant n$. Let $\eta > 0$ be such that $\eta < \min_{i \neq j}\{\operatorname{dist}(F_i, F_j)\}$. Then if $f \in C_\lambda$ and $D(f, g) < \eta$ we have for all x that $f(x) \in F_j$ if and only if $g(x) \in F_j$, and, hence, $\gamma \circ f = \gamma \circ g$, so $\|\gamma \circ f - h\|_\infty < \varepsilon$.

PROOF OF THEOREM 1. Let $f \in C_\lambda$. Then f is a homeomorphism of E into E if and only if f is one-to-one. Also, if f is not one-to-one, it is clear that there are an $h \in C(E, R)$ and $\varepsilon > 0$ such that $f \notin G(h, \varepsilon)$. It follows that f is a homeomorphism of E into E if and only if f is in every $G(h, \varepsilon)$.

Let $\{h_n\}_{n=1}^\infty$ be dense in $C(E, R)$. Then f is a homeomorphism of E into E if and only if f is in $\bigcap_{n,k=1}^\infty G(h_n, k^{-1})$. Combining this with Theorem 3 and Lemmas 2 and 3 and applying the Baire Category Theorem, we see that the homeomorphisms in C_λ form a dense G_δ subset of the complete metric space C_λ. This, together with Theorem 2, completes the proof.

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