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A NOTE ON THE DIVISIBILITY OF CERTAIN CHERN NUMBERS

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ABSTRACT. If M is a weakly almost complex manifold, then $c_r(M) \in H^{2A}(M; Z)$ is the rth Chern class of its normal bundle.

THEOREM 1. *If m, r are natural numbers with $r \leqslant m$, then there exists a $2m$-fold M_0, compact, closed and weakly almost complex, so that the normal characteristic number $\langle c_r(M_0)c_{m-r}(M_0), [M_0]\rangle$ is a power of* 2.

1. Introduction. First, let us set the terminology. In this note the word "manifold" will mean a closed, compact C^∞ differentiable weakly almost compact manifold. All cohomology groups will have integer coefficients. If M is a $2m$-fold, then we define the normal characteristic number

$$\Delta_r^m(M) := (-1)^m\langle c_r(M)c_{m-r}(M), [M]\rangle$$

for $0 \leqslant r \leqslant m$ (the index m will be suppressed from $\Delta_r^m(M)$, when it will be clear from the context).

In [4], E. Rees and E. Thomas determined the highest power of 2 which divides all the numbers $\Delta_r^m(M)$, where M ranges over all $2m$-folds. In this note we will prove that the greatest common divisor of these numbers is the power of 2 determined by Rees-Thomas. (*Note*. Actually Rees-Thomas proved this assertion for $r = 0, 1, 2$.) The question of divisibility at Chern-numbers has an interest of its own (see [2], where some calculations are done for low dimensional cases). The proof splits naturally in two cases. By elaborate computations with products of complex projective spaces we prove that there are manifolds M so that $\Delta_r^m(M)$ is not divided by 3. The same assertion for the case of prime numbers greater than 3 is based on the work of E. Brown and F. Peterson [1].

2. Computations. Our next lemma puts together a few well-known facts about characteristic numbers.

LEMMA 2. (a) *$\Delta_r^{m+n}(MN) = \sum_{i+j=r}\Delta_i^m(M)\Delta_j^n(N)$, where M is a $2m$-fold and N is a $2n$-fold.*

(b) *If P_n is the n-complex projective space, then*

$$\Delta_r^n(P_n) = \binom{n+r}{n}\binom{2n-r}{n}.$$

(*The case P_0 = point is allowed.*)

(c) *If P_1^n is the product $P_1 \times P_1 \times \cdots \times P_1$ (n times), then $\Delta_r^n(P_1^n) = \binom{n}{r}2^n$.*

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PROOF. See [4].

The next lemma is very well known.

LEMMA 3. *Let* $a = a_0 + a_1 p + \cdots + a_n p^n + \ldots, b = b_0 + b_1 p + \cdots + b_n p^n + \ldots$, *where* a, b *are natural numbers,* p *is a prime number and* $a_0, a_1, \ldots, a_n, \ldots, b_0, b_1, \ldots, b_n, \ldots = 0, 1, \ldots, (p-1)$. *Then*

$$\binom{\alpha}{\beta} \equiv \binom{a_0}{b_0}\binom{a_1}{b_1}\cdots\binom{a_n}{b_n}\cdots, \qquad \operatorname{mod} p.$$

COROLLARY 4. (a)

$$\Delta_r(P_1^{3^k}) \equiv \begin{cases} 0 \bmod 3 & \text{if } 0 < r < 3^k, \\ -1 \bmod 3 & \text{if } r = 0, 3^k. \end{cases}$$

(b)

$$\Delta_r(P_1^2 \cdot 3^k) \equiv \begin{cases} 0 \bmod 3 & \text{if } r \neq 0, 3^\mu, 2\cdot 3^k, \\ 1 \bmod 3 & \text{if } r = 0, 2\cdot 3^k, \\ -1 \bmod 3 & \text{if } r = 3^k. \end{cases}$$

(c)

$$\Delta_r(P_{3^k}) \equiv \begin{cases} 1 \bmod 3 & \text{if } 0 < r < 3^k, \\ -1 \bmod 3 & \text{if } r = 0, 3^k. \end{cases}$$

PROOF. It comes from the combination of Lemma 2, parts (b) and (c), and Lemma 3.

LEMMA 5. *Let* r, k, n *be natural numbers so that* $r < 3^k \leqslant n - r$. *Then* $\Delta_r^n(MP_1^{3^k}) \equiv -\Delta_r^{n-3^k}(M) \bmod 3$ *for all* $2(n-3^k)$*-folds* M.

PROOF. Immediate from Lemma 2(a) and Corollary 4(a).

LEMMA 6. *Let* r, k, n *be natural numbers so that* $r < 3^k \leqslant n$ *and* $(n-r) < 3^k$. *Then* $\Delta_r^n(P_{3^k}P_1^{n-3^k}) \not\equiv 0 \bmod 3$.

PROOF. From Lemma 2(a), (c) and Corollary 4(c), we have

$$\begin{aligned} \Delta_r^n(P_{3^k}P_1^{n-3^k}) &\equiv \pm \sum_{i=0}^{n-3^k} \binom{n-3^k}{i} \mod 3 \\ &\equiv \pm (1+1)^{n-3^k} \mod 3 \\ &\equiv \pm 1 \mod 3. \end{aligned}$$

LEMMA 7. *Let* r, n, k *be natural numbers so that* $3^k \leqslant n - r < 2\cdot 3^k \leqslant n$ *and* $2\cdot 3^k < r$. *Then*

$$\Delta_r^n(M\cdot P_1^{2\cdot 3^k}) \equiv -\Delta_{r-3^k}^{n-2\cdot 3^k}(M) \mod 3$$

for all $2\cdot(n-2\cdot 3^k)$*-folds* M.

PROOF. It is an easy calculation from Lemma 2(a) and Corollary 2(b).

LEMMA 8. *Let* r, n, k *be natural numbers so that* $3^k - r < n < 3^{k+1}$. *Then* $\Delta_r^n(P_1^n) \neq 0 \bmod 3$.

PROOF. It is an immediate consequence of Lemma 2(c) and Lemma 3.

3. Proof of the main result. Now we are ready to give the

PROOF OF THEOREM 1. It is enough to prove that if p is a prime number greater than 2, then there is a $2m$-fold M_0 so that $\Delta_r^m(M_0) \neq 0 \bmod p$.

We distinguish two cases, depending on whether $p = 3$ or p is greater than 3.

Case 1. Let $p = 3$. Then the result follows by induction on n, using Lemmas 5, 6, 7, 8 combined with the obvious remark that we can interchange the roles of r and $(m - r)$ because, clearly, $\Delta_r^m = \Delta_{m-r}^m$. If $m = 0$ our assertion is obvious. In the other cases Lemma 5 and 7 permit us to reduce our assertion to a lower m, with two exceptions, which are covered by Lemmas 6 and 8.

Case 2. Let $p > 3$. Let A be the subalgebra of the mod-p Steenrod algebra, generated by the reduced powers $P^1, P^2, \ldots$. J. Milnor, in [**3**, Theorem 2], proved that $H^*(MU; Z_p)$ is a free A-module, and a free basis is determined by the elements of the form

$$s(a_1, a_2, \ldots, a_k)U = \left(\sum t_1^{a_1} t_2^{a_2} \cdots t_k^{a_k}\right)U,$$

where none of the integers $a_1, a_2, \ldots, a_k$ is of the form $p^i - 1$. It is easy to observe that if $c_a, c_b \in H^*(BU; Z_p)$ are the mod-p Chern classes defined by $c_a = \sum t_1 \cdots t_a$ and $c_b = \sum t_1 \cdots t_b$, then $c_a c_b = (\sum t_1 \cdots t_a)(\sum t_1 \cdots t_b)$, so $c_a c_b U$ is a sum of elements of Milnor's basis described above.

This is not true for $p = 3$, because in this case the number 2 is of the form $(p^i - 1)$.

On the other hand, Brown and Peterson, in [**1**, Theorem 1.2], gave criteria when a mod-p characteristic class $x \in H^n(BU; Z_p)$ is zero on all $2n$-folds (meaning as a normal characteristic number) and the criterion is the following: If $\{x_i\}_{i \in I}$ is an A-basis for $H^*(MU; Z_p)$, then there must be a $a_i \in A$ for $i \in I$ with $\deg a_i > 0$ so that $xU = \sum_{i \in I} a_i x_i$. Of course Brown and Peterson deal with tangential characteristic numbers, but their proof works the same (it is even simpler) with normal characteristic numbers.

From the remarks above, clearly $c_a c_b U$ cannot meet this criterion.

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