PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 84, Number 2, February 1982

ON FINITELY DOMINATED CW COMPLEXES

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ABSTRACT. Let $\mathfrak{D}$ be the class of all CW complexes homotopy dominated by finite CW complexes. In this paper we prove the following theorem.

THEOREM. *Suppose a connected CW complex $X \in \mathfrak{D}$ is a union of two connected subcomplexes X_1, X_2 with $X_1 \cap X_2 = X_0 \in \mathfrak{D}$. Then $X_1, X_2 \in \mathfrak{D}$ if one of the following conditions is satisfied*:

(i) *$\pi_1(X_0, x) \to \pi_1(X, x)$ is a monomorphism for each $x \in X_0$*,

(ii) *$\pi_1(X_i) \to \pi_1(X)$ is a monomorphism for $i = 1, 2$ and $\pi_1(X_1)$, $\pi_1(X_2)$ are finitely presented.*

1. Introduction. Let $\mathfrak{D}$ be the class of all CW complexes homotopy dominated by finite CW complexes. In this note we discuss the following question: Suppose a connected CW complex X is a union of two connected subcomplexes X_1 and X_2. Under what conditions does $X \in \mathfrak{D}$ and $X_1 \cap X_2 = X_0 \in \mathfrak{D}$ imply that X_1, $X_2 \in \mathfrak{D}$?

In [4, p. 48] L. C. Siebenmann answered the above question positively in case where $\pi_1(X_i) \to \pi_1(X)$ has a left inverse for $i = 1, 2$ and asked (p. 49) whether the condition that $\pi_1(X_i) \to \pi_1(X)$ is a monomorphism, $i = 1, 2$, is sufficient for X_1 and X_2 to be in $\mathfrak{D}$.

The following results give partial answers to Siebenmann's question.

1.1. THEOREM. *Suppose a connected CW complex $X \in \mathfrak{D}$ is a union of two connected subcomplexes X_1 and X_2 with $X_1 \cap X_2 = X_0 \in \mathfrak{D}$. If, for $i = 1, 2$,*

(i) *$\pi_1(X_i) \to \pi_1(X)$ is a monomorphism and*

(ii) *$\pi_1(X_i)$ is finitely presented,*

then $X_1, X_2 \in \mathfrak{D}$.

1.2. THEOREM. *Let $X \in \mathfrak{D}$ satisfy the hypotheses of Theorem* 1.1. *If $\pi_1(X_1) \to \pi_1(X)$ is a monomorphism, $\pi_1(X_1)$ is finitely presented and $X_0, X_2 \in \mathfrak{D}$, then $X_1 \in \mathfrak{D}$.*

1.3. THEOREM. *Let $X \in \mathfrak{D}$ satisfy the hypotheses of Theorem* 1.1. *If $\pi_1(X_0) \to \pi_1(X)$ is a monomorphism, then conditions* (i) *and* (ii) *of Theorem* 1.1 *hold. In particular, $X_1, X_2 \in \mathfrak{D}$.*

Examples in [4, pp. 49, 83–89] show that some restriction on fundamental groups is necessary.

Received by the editors December 16, 1980.

1980 *Mathematics Subject Classification*. Primary 55P15; Secondary 54E60, 20E06.

Key words and phrases. Finitely dominated CW complexes, free products with amalgamation.

0002-9939/82/0000-0424/$02.25

2. Proof of Theorem 1.1. We need the following proposition.

2.1. PROPOSITION. *Suppose G is a subgroup of a group H and A is a G module. If $Z[H] \otimes_G A$ is a projective (finitely generated) H module, where for the tensor product $Z[H]$ has the right G module structure given by $G \hookrightarrow H$, then A is projective (finitely generated).*

PROOF (DUE TO THE REFEREE). The statement about projective modules is obvious. Although the part concerning finite generation is well known, no convenient proof is in the literature and a proof is therefore given.

Since G is a subgroup of H, $Z[H]$ is a free right $Z[G]$ module. In fact, if $\{t_\lambda \in H | \lambda \in \Lambda\}$ is a transversal (i.e. a set of H/G coset representatives) for G in H, then $\{t_\lambda | \lambda \in \Lambda\}$ is a $Z[G]$ basis for $Z[H]$. Without loss of generality, we may assume that the identity coset $1G$ is represented by $1 \in H$. It follows easily that if M is any left $Z[G]$ module, then, as $Z[G]$ modules, $Z[H] \otimes_G M = \Sigma_{\lambda \in \Lambda} t_\lambda M$ where one $t_\lambda = 1$ (i.e. M is a $Z[G]$ direct summand of $Z[H] \otimes_G M$).

Now, let $a_1, \ldots, a_k \in Z[H] \otimes_G A$ be a finite set of $Z[H]$ generators. Then each a_i can be written as a tuple $a_i = (a_i^\lambda)$ $(\lambda \in \Lambda)$, where $a_i^\lambda \in t_\lambda A$ for all λ. In fact, $a_i^\lambda = t_\lambda b_i^\lambda$, for some $b_i^\lambda \in A$ and $b_i^\lambda \neq 0$, for only finitely many λ. Let B be the $Z[G]$ submodule of A generated by $\{b_i^\lambda | i = 1, \ldots, k, \lambda \in \Lambda\}$. Then B is finitely generated and the inclusion $i\colon B \to A$ induces an isomorphism $1 \otimes i\colon Z[H] \otimes_G B \to Z[H] \otimes_G A$. Since

$$Z[H] \otimes_G B \xrightarrow{1\otimes i} Z[H] \otimes_G A \to Z[H] \otimes_G (A/B) \to 0$$

is exact, it follows that $Z[H] \otimes_G (A/B) = 0$. Since A/B is a direct summand of this module, it follows that $A/B = 0$ and that A is finitely generated.

Now, the proof of Theorem 1.1 is divided into three cases.

Case 1. Assume: 1. X_0 is connected, 2. X_0 is homotopy equivalent to a finite CW complex K_0 and 3. X is homotopy equivalent to a finite CW complex K.

Let $f_0\colon K_0 \to X_0$ be a homotopy equivalence. For each $i = 1, 2$ we can extend f_0 to $f_i'\colon K_i' \to X_i$ such that f_i' induces an epimorphism of fundamental groups and K_i' is the wedge of K_0 and a finite number of circles. By Lemma 3.11 in [4, p. 18], the kernel of $\pi_1(f_i')$ can be expressed as the normal closure of a finite set of elements of $\pi_1(K_i')$. Therefore, by attaching a finite number of 2-cells to K_i' we can form a finite CW complex K_i and an extension $f_i\colon K_i \to X_i$ of f_0, inducing an isomorphism of fundamental groups. Thus the following proposition P_m holds for $m = 1$.

P_m: There exists a finite CW complex L_m with $\dim L_m \leqslant \max(m, 2 + \dim K_0)$ that is a union of subcomplexes K_1 and K_2 with intersection equal to K_0, and a map $f\colon L_m \to X$ so that, restricted to K_k, $k = 0, 1, 2$, f gives a map $f_k\colon K_k \to X_k$ which is m-connected for $k = 1, 2$, a homotopy equivalence for $k = 0$ and each $\pi_1(f_k)$, $k = 1, 2$, is an isomorphism.

Suppose for induction that P_{m-1} holds, $m \geqslant 2$, and consider the exact sequence

$$0 \to C_*(\overline{M}(f_0), \overline{K}_0) \to C_*(\overline{M}(f_1), \overline{K}_1) \oplus C_*(\overline{M}(f_2), \overline{K}_2) \to C_*(\tilde{M}(f), \tilde{L}_{m-1}) \to 0,$$

where $\bar{S} = p^{-1}(S)$, $p\colon \tilde{M}(f) \to M(f)$ being the universal covering projection of the mapping cylinder $M(f)$ of $f\colon L_{m-1} \to X$ and $C_*(Y, Z)$ the chain complex of a CW pair (Y, Z).

Since $f_0\colon K_0 \to X_0$ is a homotopy equivalence, so is $\bar{K}_0 \to \bar{M}(f_0)$. Thus, $H_*(\bar{M}(f_0), \bar{K}_0) = 0$ and we have an isomorphism of $\pi_1(X)$ modules $H_n(\bar{M}(f_1), \bar{K}_1) \oplus H_n(\bar{M}(f_2), \bar{K}_2)$ and $H_n(\tilde{M}(f), \tilde{L}_{m-1})$ for $n \geq 0$. $H_m(\tilde{M}(f), \tilde{L}_{m-1})$ is the first nonvanishing homology group of $(\tilde{M}(f), \tilde{L}_{m-1})$ and, by Lemma 4.6 in [4] (see also [5, Theorem A]), it is finitely generated. Hence, both $H_m(\bar{M}(f_i), \bar{K}_i)$, $i = 1, 2$, are finitely generated $\pi_1(X)$ modules.

It is well known (see [4, Lemma 6.7]) that

$$C_*\big(\bar{M}(f_i), \bar{K}_i\big) = Z[\pi_1(X)] \otimes C_*(\tilde{M}(f_i), \tilde{K}_i)$$

where the tensor product is over $Z[\pi_1(X_i)]$ and $\tilde{M}(f_i)$, $\tilde{K}_i$ denote universal covers. Since $\pi_1(X_i) \to \pi_1(X)$ is monomorphic, $Z[\pi_1(X)]$ is a free $Z[\pi_1(X_i)]$ module and thus tensoring with it is an exact functor. Hence,

$$H_m\big(\bar{M}(f_i), \bar{K}_i\big) = Z[\pi_1(X)] \otimes H_m(\tilde{M}(f_i), \tilde{K}_i)$$

where again the tensor product is over $Z[\pi_1(X_i)]$. By Proposition 2.1, $H_m(\tilde{M}(f_i), \tilde{K}_i)$ is a finitely generated $\pi_1(X_i)$ module, $i = 1, 2$. As in [5] (see the proof of Theorem A) we can obtain L_m and $f'\colon L_m \to X$ by attaching a finite number of m-cells to K_1 and K_2 and then extending f onto L_m. This completes the induction.

The proof that $X_1, X_2 \in \mathfrak{D}$ is completed as follows. Take an $(m-1)$-connected map $f\colon L \to X$ as in P_{m-1}, where $m \geq 1 + \max(\dim K_0, \dim K, 2)$. Then $H_*(\tilde{M}(f), \tilde{L})$ is finitely generated, projective and concentrated in dimension m (see [5, Theorem A and Lemma 2.1]). Since

$$H_n\big(\bar{M}(f_1), \bar{K}_1\big) \oplus H_n\big(\bar{M}(f_2), \bar{K}_2\big) \simeq H_n(\tilde{M}(f), \tilde{L}),$$

for each n, we infer that $H_*(\bar{M}(f_i), \bar{K}_i)$ is finitely generated, projective over $Z[\pi_1(X)]$ and concentrated in dimension m for $i = 1, 2$. Hence, $H_*(\tilde{M}(f_i), \tilde{K}_i)$ is a finitely generated projective over $Z[\pi_1(X_i)]$ and concentrated in dimension m (by Proposition 2.1). It follows from [4, Lemma 6.2] that $X_1, X_2 \in \mathfrak{D}$.

Case 2. Assume X_0 is connected. Then, $X \times S^1$ and $X_0 \times S^1$ have the homotopy type of finite complexes (see [2]) and by Case 1 we have, $X_1 \times S^1, X_2 \times S^1 \in \mathfrak{D}$, which implies $X_1, X_2 \in \mathfrak{D}$.

General case. If X_0 is not connected, we attach a finite number of 1-cells $e_1, \dots, e_n$ to it so that $X_0' = X_0 \cup e_1 \cup \cdots \cup e_n$ is connected. By Case 2, both $X_1' = X_1 \cup X_0'$ and $X_2' = X_2 \cup X_0'$ belong to $\mathfrak{D}$. Since X_i is a retract of X_i', $i = 1, 2$, we infer $X_1, X_2 \in \mathfrak{D}$.

Remark. Our proof of Theorem 1.1 is a simplified version of the original Siebenmann's proof of Complement 6.6(b) (see [4, pp. 48, 54–56]).

Theorem 1.2 can be proved similarly.

3. Proof of Theorem 1.3. Theorem 1.3 is basically an algebra theorem and will be derived from the following result.

3.1. THEOREM. *If the free product* $P = \langle G * H\colon A = B, \varphi\rangle$ *with amalgamation is finitely presented and* A *is finitely presented, then both* G *and* H *are finitely presented.*

For the notion of the free product $P = \langle G * H\colon A = B, \varphi\rangle$ (denoted also by $G *_A H$) of groups G and H, amalgamating subgroups A of G and B of H by an isomorphism φ, see [**1**, p. 179].

The proof of this theorem depends on two lemmata.

3.2. LEMMA. *If* $G *_A H$ *is finitely generated, so are* G *and* H.

PROOF. Let $\{g_\lambda | \lambda \in L\}$ and $\{h_\mu | \mu \in M\}$ be generators for G and H respectively. By [**1**, p. 187], $\{g_\lambda, h_\mu | \lambda \in L, \mu \in M\}$ generate $G *_A H$. Since this group is finitely generated, there exist finitely many of the g_λ's and h_μ's, $g_1, \dots, g_q, h_1, \dots, h_r$ such that these elements generate $G *_A H$. But then $g_1, \dots, g_q$ generate G and $h_1, \dots, h_r$ generate H.

Suppose now that A has presentation

$$\langle Z_1, \dots, Z_s;\ W_1(Z_1, \dots, Z_s), \dots, W_p(Z_1, \dots, Z_s)\rangle.$$

Then G has a presentation

$$\langle X_1, \dots, X_q, Z_1, \dots, Z_s;\ U_\lambda(X_i, Z_k), W_1(Z_1, \dots, Z_s), \dots, W_p(Z_1, \dots, Z_s)\rangle,$$

where $\lambda \in L$, $U_\lambda(X_i, Z_k)$ is a word in (possibly all) the X_i's and Z_k's, and $\{X_i\}$ maps to a finite set of generators $\{g_i\}$ for G. Suppose that H is presented similarly as

$$\langle Y_1, \dots, Y_r, Z_1, \dots, Z_s;\ V_\mu(Y_j, Z_l), W_1(Z_1, \dots, Z_s), W_p(Z_1, \dots, Z_s)\rangle,$$

where $\mu \in N$.

3.3. LEMMA. $G *_A H$ *has the presentation*

$$\langle X_1, \dots, X_q, Y_1, \dots, Y_r, Z_1, \dots, Z_s;\ U_\lambda(X_i, Z_k), V_\mu(X_j, Z_l), \dots, W_p(Z_1, \dots, Z_s)\rangle.$$

PROOF. This is well known (see [**3**]).

PROOF OF THEOREM 3.1. We present A, G, H as above and consider the presentation of $G *_A H$ given by 3.3. Since $G *_A H$ is finitely presented, it has a presentation on the above generators containing only finitely many of the above relators. Thus $G *_A H$ may be presented as

$$\langle X_1, \dots, X_q, Y_1, \dots, Y_r, Z_1, \dots, Z_s;\ U_1(X_i, Z_k), \dots, U_m(X_i, Z_k), V_1(Y_j, Z_l), \dots, V_n(Y_j, Z_l), W_1(Z_1, \dots, Z_s), \dots, W_p(Z_1, \dots, Z_s)\rangle.$$

Let

$$G' = \langle X_1, \dots, X_q, Z_1, \dots, Z_s;\ U_1(X_i, Z_k), \dots, U_m(X_i, Z_k), W_1(Z_1, \dots, Z_s), \dots, W_p(Z_1, \dots, Z_s)\rangle.$$

Then there is an obvious epimorphism ψ': $G' \to G$ that maps the subgroup $A' \subset G'$ generated by $Z_1, \ldots, Z_s$ isomorphically onto A. Similarly, if $H' = \langle Y_1, \ldots, Y_r, Z_1, \ldots, Z_s; V_1(Y_j, Z_1), \ldots, V_n(Y_j, Z_1), W_1(Z_1, \ldots, Z_s), W_p(Z_1, \ldots, Z_s)\rangle$, there is an epimorphism ψ'': $H' \to H$ that maps the subgroup generated by $Z_1, \ldots, Z_s$ isomorphically onto A. Thus ψ' and ψ'' induce a homomorphism ψ: $G' *_A H' \to G *_A H$. Since the presentation for $G' *_A H'$ is identical with that of $G *_A H$, ψ is an isomorphism. It follows easily that ψ' and ψ'' are isomorphisms and, therefore, that G and H are finitely presented.

PROOF OF THEOREM 1.3. Suppose X_0 is connected. Then $\pi_1(X)$ is the free product of $\pi_1(X_1)$ and $\pi_1(X_2)$ with amalgamation of

$$\operatorname{im}(\pi_1(X_0) \to \pi_1(X_1)) \quad \text{and} \quad \operatorname{im}(\pi_1(X_0) \to \pi_1(X_2))$$

(see [**1**, p. 180]). Therefore $\pi_1(X_i) \to \pi_1(X)$ is a monomorphism for $i = 1, 2$ and, by Theorem 3.1, $\pi_1(X_1)$ and $\pi_1(X_2)$ are finitely presented.

By Theorem 1.1, $X_1, X_2 \in \mathfrak{D}$.

If X_0 is not connected, then we apply the trick used in General Case of the proof of Theorem 1.1.

I am indebted to the referee for supplying me with the proof of Proposition 2.1 and for simplifying the proof of Theorem 3.1.

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