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A LOCALLY CLOSED SET WITH A SMOOTH GROUP STRUCTURE IS A LIE GROUP

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ABSTRACT. We prove the following result. Let V be a smooth manifold and let $G \subset V$ be a locally closed set with a group structure such that both multiplication and inversion are smooth maps; then G is an imbedded smooth submanifold of V. This result is a generalization of the well-known fact that a closed subgroup of a Lie group is itself a Lie group, because we are not assuming any group structure in the manifold V.

1. Introduction. Our aim is to prove the following result.

THEOREM. *Let V be a finite-dimensional manifold of class C^{p+1} $(1 \leqslant p \leqslant +\infty)$ and let $G \subset V$ be a locally closed set with a group structure, such that the multiplication π: $G \times G \to G$ is of class C^{p+1} and the map ξ: $G \to G$, which takes each element into its inverse, is of class C^1. Then G is an imbedded submanifold of class C^p of V.*

We present in the next paragraph the tools used in the proof of the theorem, and the third paragraph will consist of this proof.

2. Tools and notations. The word *manifold* will mean always a finite-dimensional boundaryless manifold, not necessarily paracompact or Hausdorff. By *submanifold* we mean always an imbedded submanifold.

Let V and V' be manifolds of classs C^p. If $A \subset V$ is a set, we say that a map f: $A \to V'$ is *of class C^p* if, for each $a \in A$, there is an open neighbourhood U of a in V and a C^p map $\bar{f}$: $U \to V'$ such that f and $\bar{f}$ agree on $A \cap U$ (we call $\bar{f}$ a *local extension* of f). Equivalently, one may replace V and V' in the definition by two C^p submanifolds V_0 and V_0', such that $A \subset V_0$ and $f(A) \subset V_0'$. As usual, the composite of two C^p maps is again a C^p map. If $A \subset V$ and $A' \subset V'$ a bijective map f: $A \to A'$ is called a C^p *diffeomorphism* if both f and f^{-1} are of class C^p. In the case where $A \subset \mathbf{R}^N$ and f: $A \to \mathbf{R}^{N'}$ is a map, a standard partition of unity argument shows that f is of class C^p if and only if there is an open set U containing A and a global extension $\bar{f}$: $U \to \mathbf{R}^{N'}$ of f of class C^p.

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If $A \subset \mathbf{R}^N$ is a set and if $a \in A$, we say that $h \in \mathbf{R}^N$ is *strictly tangent to A at a* (Bouligand [2]) if there exists a sequence $x_n \in A$, with $x_n \to a$, and a sequence $t_n \in \mathbf{R}$, with $t_n > 0$, such that $t_n(x_n - a) \to h$. We will denote by $t_a(A)$ the set whose elements are the strictly tangent vectors to A at a; it is not in general a vector space in $\mathbf{R}^N$, but it is always a closed cone that contains 0. We will denote by $T_a(A)$ the vector space spanned by $t_a(A)$ and its elements will be called *tangent vectors to A at a*. These notions are local, in the obvious sense, and, if $A \subset \mathbf{R}^N$ is an open set, $t_a(A) = T_a(A) = \mathbf{R}^N$.

If one uses the definition of the derivative of a map, one proves very easily the following two propositions.

2.1. Proposition. *Let $A \subset \mathbf{R}^N$ and f: $A \to \mathbf{R}^{N'}$ be a C^1 map. If $a \in A$ and if $\bar{f}$: $U \to \mathbf{R}^{N'}$ and $\bar{f}'$: $U' \to \mathbf{R}^{N'}$ are two C^1 local extensions of f whose domains contain a, then the linear maps $D\bar{f}(a)$ and $D\bar{f}'(a)$, from $\mathbf{R}^N$ into $\mathbf{R}^{N'}$, agree on $T_a(A)$.*

This proposition allows us to consider a well-defined linear map $Df(a)$: $T_a(A) \to \mathbf{R}^{N'}$, which we call the *derivative* of f at a. The second proposition shows that this notion of derivative is functorial.

2.2. Proposition. *Let $A \subset \mathbf{R}^N$, $A' \subset \mathbf{R}^{N'}$ and f: $A \to A'$ be a C^1 map. Then, for each $a \in A$, $Df(a)$ maps $T_a(A)$ into $T_{f(a)}(A')$ and $t_a(A)$ into $t_{f(a)}(A')$.*

The functoriality is completed by the chain rule, which follows trivially from the usual chain rule for maps defined on open sets. By a functorial argument, the following proposition follows.

2.3. Proposition. *Let $A \subset \mathbf{R}^N$, $A' \subset \mathbf{R}^{N'}$ and f: $A \to A'$ be a C^1 diffeomorphism. Then, for each $a \in A$, $Df(a)$ is an isomorphism from $T_a(A)$ onto $T_{f(a)}(A')$, and maps $t_a(A)$ onto $t_{f(a)}(A')$.*

We extend now in a natural way the definition of $t_a(A)$ and $T_a(A)$ to the case where V is a C^1 manifold and $A \subset V$. Choose in fact a local parametrization φ: $W \to U$, where W is open in $\mathbf{R}^N$ and U is an open neighbourhood of a in V and let $\varphi(b) = a$. Then $D\varphi(b)$ is an isomorphism from $\mathbf{R}^N$ onto $T_a(V)$ and we define $t_a(A)$ and $T_a(A)$ to be the images by this isomorphism of $t_b(\varphi^{-1}(A \cap U))$ and $T_b(\varphi^{-1}(A \cap U))$. Using Proposition 2.3, this definition is seen to be independent of the parametrization we have chosen. One sees easily that these definitions remain equivalent if instead of V one uses a submanifold V_0 such that $A \subset V_0$. It is now straightforward to prove the analogue of Proposition 2.1, with two C^1 manifolds V and V' instead of $\mathbf{R}^N$ and $\mathbf{R}^{N'}$, which enables us to define $Df(a)$ as a linear map from $T_a(A)$ into $T_{f(a)}(V')$, and to prove then the analogues of Propositions 2.2 and 2.3. We will also refer to these generalizations as Propositions 2.2 and 2.3.

Let V and V' be C^1 manifolds and let $a \in A \subset V$ and $a' \in A' \subset V'$. Applying Propositions 2.2 and 2.3 to the canonical projections $A \times A' \to A$ and $A \times A' \to A'$ and to the canonical injections $A \to A \times A'$ and $A' \to A \times A'$, we get the following.

2.4. Proposition. *We have*

$$T_{(a,a')}(A \times A') = T_a(A) \times T_{a'}(A'), \qquad t_{(a,a')}(A \times A') \subset t_a(A) \times t_{a'}(A').$$

We will also use tangent fields defined on a set A contained in a manifold V. Suppose that V is a manifold of class C^{p+1}, that $A \subset V$ and that X: $A \to$ is a *vector field*, i.e. X is a map such that $X(x) \in T_x(V)$ for each $x \in A$. We say that X is of class C^p if it happens so to the corresponding map from $A \subset V$ to the tangent bundle $T(V)$ of V.

We pass now to the only nontrivial result of this paragraph. For simplicity, and because it is the only case we will have to use, we will quote only the $\mathbf{R}^N$-version of this result.

Suppose that $U \subset \mathbf{R}^N$ is an open set and that X: $U \to \mathbf{R}^N$ is a C^p map, where $1 \leqslant p \leqslant +\infty$. We can then take the *flow* of X, which is a C^p map F: $\Omega \to U$, with Ω open in $\mathbf{R} \times U$ and containing $\{0\} \times U$, such that

$$F(0, x) = x, \qquad (\partial F/\partial t)(t, x) = X(F(t, x)).$$

2.5. THEOREM (BRÉZIS [3]). *Let $U \subset \mathbf{R}^N$ be an open set and X: $U \to \mathbf{R}^N$ be a C^1 map, and let F: $\Omega \to U$ be its flow. Let A be a closed set in U such that, for each $x \in A$, $X(x) \in t_x(A)$. If $(t, x) \in \Omega$, with $x \in A$ and $t \geqslant 0$, then $F(t, x) \in A$.*

In fact, the result proved by Brézis is the theorem stated above, but with a stronger notion of strictly tangent vector that was not suitable for our purposes. A proof of this result, with the notion of strictly tangent vector we are using, can be found in [4] and is just a sharpening of Brézis's proof.

The conclusion of Theorem 2.5 can be stated by saying that the set A is flow-invariant to the right side. Applying the result to the vector field $-X$, we have

2.6. COROLLARY. *Let $U \subset \mathbf{R}^N$ be an open set and X: $U \to \mathbf{R}^N$ be a C^1 map, and let F: $\Omega \to U$ be its flow. Let A be a closed set in U such that, for each $x \in A$, $-X(x) \in t_x(A)$. If $(t, x) \in \Omega$, with $x \in A$ and $t \leqslant 0$, then $F(t, x) \in A$.*

3. Proof of the theorem. The main part of the proof will consist of the following lemma.

3.1. LEMMA. *Let V be a C^{p+1} manifold, with $1 \leqslant p \leqslant +\infty$. Let $A \subset V$ be a locally closed set and, for each $1 \leqslant i \leqslant k$, let X_i: $A \to$ be a vector field of class C^p such that, for each $x \in A$, $X_i(x) \in t_x(A)$ and $-X_i(x) \in t_x(A)$. Suppose that $a \in A$ is such that the vectors $h_i = X_i(a)$, with $1 \leqslant i \leqslant k$, are a basis for $T_a(A)$. Then there exists an open neighbourhood W of a in A such that W is a k-dimensional C^p submanifold of V.*

PROOF OF THE LEMMA. The result being local, we can suppose that V is $\mathbf{R}^N$. Then each X_i is a C^p map from A into $\mathbf{R}^N$ and we can fix an open set U of $\mathbf{R}^N$ containing A and, for each $1 \leqslant i \leqslant k$, a C^p extension $\overline{X}_i$: $U \to \mathbf{R}^N$ of X_i. The fact that A is locally closed allows us to suppose that A is closed in U, taking eventually a smaller U.

For each $1 \leqslant i \leqslant k$, let F_i: $\Omega_i \to U$ be the flow of $\overline{X}_i$. We know that F_i is a C^p defined in an open set Ω_i in $\mathbf{R} \times U$ which contains $\{0\} \times U$. The result of Brézis and its corollary (Theorem 2.5 and Corollary 2.6) show us that if $(t, x) \in \Omega_i$ and $x \in A$, then $F_i(t, x) \in A$.

Fix $\varepsilon > 0$ such that we have a well-defined C^p map F: $]-\varepsilon, \varepsilon[^k \times B_\varepsilon(a) \to U$,

$$F(t_1, \ldots, t_k, x) = F_k(t_k, F_{k-1}(t_{k-1}, \ldots, F_1(t_1, x), \ldots)),$$

where $B_\varepsilon(a)$ is the open ball. Fix $r > 0$ such that we have a well-defined C^p map φ: $]-r, r[^k \to A$,

$$\varphi(t_1, \ldots, t_k) = F_1(t_1, F_2(t_2, \ldots, F_k(t_k, a), \ldots)).$$

We have $\varphi(0, \ldots, t_i, \ldots, 0) = F_i(t_i, a)$; hence

$$(\partial\varphi/\partial t_i)(0, \ldots, 0) = X_i(a) = h_i.$$

We now choose vectors $h_{k+1}, \ldots, h_N$ in $\mathbf{R}^N$ such that the h_i, with $1 \leqslant i \leqslant N$, are a basis for $\mathbf{R}^N$, and we define a C^p map ψ: $]-r, r[^N \to \mathbf{R}^N$,

$$\psi(t_1, \ldots, t_N) = \varphi(t_1, \ldots, t_k) + t_{k+1}h_{k+1} + \cdots + t_N h_N.$$

We have $\psi(0) = a$ and, for $1 \leqslant i \leqslant N$, $(\partial\psi/\partial t_i)(0, \ldots, 0) = h_i$, so that by the inverse function theorem we conclude that, taking a smaller r, ψ is a diffeomorphism of $]-r, r[^N$ onto an open subset U' of $\mathbf{R}^N$. We can also suppose that $r \leqslant \varepsilon$ and that $U' \subset B_\varepsilon(a)$.

We will now prove that if r is small enough, then $\psi(t_1, \ldots, t_N) \in A$ implies $t_{k+1} = t_{k+2} = \cdots = t_N = 0$, which will prove the lemma, with $W = U' \cap A$.

Suppose that this was not the case and let us try to get a contradiction.

We can find sequences $t_{1,n} \to 0, t_{2,n} \to 0, \ldots, t_{N,n} \to 0$, such that $\psi(t_{1,n}, \ldots, t_{N,n}) \in A$ and, for each n, at least one of the $t_{i,n}$, with $i \geqslant k + 1$, is nonzero. Let

$$x_n = \varphi(t_{1,n}, \ldots, t_{k,n}) \in A, \qquad y_n = \psi(t_{1,n}, \ldots, t_{N,n}) \in A,$$

so that each $y_n - x_n$ is a nonzero vector of the vector space E spanned by the h_i, with $i \geqslant k + 1$. If necessary taking a subsequence, we will suppose that $\|y_n - x_n\|^{-1}(y_n - x_n)$ converges to a norm-one vector $z \in E$. We will prove that $z \in t_a(A)$, and this will be the contradiction. Remark that we have

$$a = F(-t_{1,n}, \ldots, -t_{k,n}, x_n)$$

and let

$$z_n = F(-t_{1,n}, \ldots, -t_{k,n}, y_n) \in A.$$

Let $\delta > 0$. The fact that F is a C^1 map allows us to fix $0 < \varepsilon' < \varepsilon$ such that if $|t_i| < \varepsilon'$ and $x, y \in B_{\varepsilon'}(a)$, then

$$\|DF(t_1, \ldots, t_k, x) - DF(0, \ldots, 0, a)\| \leqslant \delta.$$

Remarking that $F(0, \ldots, 0, x) = x$, we see that, for each $h \in \mathbf{R}^N$, $DF(0, \ldots, 0, a)(0, \ldots, 0, h) = h$. Hence, using the second mean value theorem, if $|t_i| < \varepsilon'$ and $x, y \in B_{\varepsilon'}(a)$, then

$$\|F(t_1, \ldots, t_k, y) - F(t_1, \ldots, t_k, x) - (y - x)\| \leqslant \delta\|y - x\|.$$

Specializing, we get, for n large enough,

$$\|z_n - a - (y_n - x_n)\| \leqslant \delta\|y_n - x_n\|,$$

$$\| \|y_n - x_n\|^{-1}(z_n - a) - \|y_n - x_n\|^{-1}(y_n - x_n)\| \leqslant \delta,$$

so that

$$\| \|y_n - x_n\|^{-1}(z_n - a) - \|y_n - x_n\|^{-1}(y_n - x_n)\| \to 0$$

and

$$\|y_n - x_n\|^{-1}(z_n - a) \to z,$$

which by definition means that $z \in t_a(A)$, as we wanted.

REMARK. J. Eells called my attention to a paper of Stefan [6]. If one uses the results of this paper, one can give a shorter, although much less elementary, proof of the preceding lemma which would use in any case the theorem of Brézis.

PROOF OF THE THEOREM STATED IN THE INTRODUCTION. Let e be the unit element of the group G and let $h_1, \ldots, h_k$ be a basis for $T_e(G)$, such that each h_i belongs to $t_e(G)$. For each $x \in G$, let L_x: $G \to G$ be the map defined by $L_x(y) = x \cdot y = \pi(x, y)$; it is a C^{p+1} map, and, in fact, a C^{p+1} diffeomorphism, its inverse being $L_{x^{-1}}$: $G \to G$. For each $1 \leq i \leq k$, we define a vector field X_i: $G \to$ by $X_i(x) = DL_x(e)(h_i)$; in fact, we even have $X_i(x) \in t_x(G)$. Using the fact that $DL_x(e)$: $T_e(G) \to T_x(G)$ is an isomorphism, it follows that, for each $x \in G$, the vectors $X_i(x)$, with $1 \leq i \leq k$, are a basis for $T_x(G)$. Each X_i is a C^p vector field, because we have $X_i(x) = D\pi(x, e)(0, h_i)$. If we can prove that, for each $x \in G$, $-X_i(x) \in t_x(G)$, we can apply the preceding lemma and we get our theorem. For this it will be enough to prove that $-h_i \in t_e(G)$ and that is what we are going to do now. First, we remark that $D\pi(e, e)(h, k) = h + k$. In fact, from $\pi(x, e) = x$, we get $D\pi(e, e)(h, 0) = h$, and from $\pi(e, y) = y$, we get $D\pi(e, e)(0, k) = k$; hence

$$D\pi(e, e)(h, k) = D\pi(e, e)(h, 0) + D\pi(e, e)(0, k) = h + k.$$

Next we remark that $D\xi(e)(h) = -h$. In fact, from $e = \pi(x, \xi(x))$, we get

$$0 = D\pi(e, e)(h, D\xi(e)(h)) = h + D\xi(e)(h).$$

Now, as ξ is a C^1 map from G into G, we conclude that $-h_i = D\xi(e)(h_i) \in t_e(G)$ as we wanted.

REMARK. Perhaps the natural framework for the theorem that we have proved is Aronszajn's notion of subcartesian space (see [1] or [5]). In fact, we could state the theorem as follows. If G is a C^{p+1} subcartesian locally compact space, with a group structure whose multiplication is C^{p+1} and whose inversion is C^1, then G is a C^p manifold. The proof would be essentially the same, but in the analogue of our Lemma 3.1, one would have to take a notion of tangent vector that is different from the one used by these authors. The advantage of this framework is that we do not need any ambient manifold V.

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