TRIANGULARIZABLE ALGEBRAS OF COMPACT OPERATORS

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ABSTRACT. It is shown that a closed algebra A of compact operators is triangularizable if and only if the algebra $A/\text{rad}(A)$ is commutative.

An algebra A of operators on a Banach space X is triangularizable if there exists a maximal chain of subspaces of X invariant for A [7]. If X is finite dimensional, this reduces to the usual concept, since in this case A is triangularizable if and only if there exists a basis for X with respect to which all the operators in A have upper triangular matrices. Recently the concept of triangularizability has attracted some attention; see [2, 6, 7]. Also, for some earlier finite dimensional results see McCoy [9, 10]. In this paper we show that if A is a closed algebra of compact operators, then A is triangularizable if and only if $A/\text{rad}(A)$ is commutative (where $\text{rad}(A)$ denotes the Jacobson radical of A). It follows that if A is radical or commutative then A is necessarily triangularizable.

NOTATION. X will always denote a complex Banach space, $\mathcal{B}(X)$ the algebra of bounded linear operators on X , and $\mathcal{K}(X)$ the ideal of compact operators. The spectrum and spectral radius are denoted by σ and r respectively. Subspaces of X are always assumed to be closed. A always denotes a complex Banach algebra. If $A \subseteq \mathcal{B}(X)$, a subspace M of X is *invariant* for A if $T(M) \subseteq M$ ($T \in A$). Finally if $S, T \in \mathcal{B}(X)$, $[S, T]$ denotes their commutator, $[S, T] = ST - TS$.

DEFINITION. A closed subalgebra A of $\mathcal{B}(X)$ is *triangularizable* if there exists a maximal chain of subspaces of X each of which is invariant for A .

Before stating the next lemma we need to establish some notation. Let $T \in \mathcal{B}(X)$, and $\mathcal{F}$ a maximal chain of subspaces, each invariant for T . If $M \in \mathcal{F}$ and $M_- \neq M$, then we can choose $z \in M$ so that $M_- \oplus [z] = M$, where $[z]$ denotes the span of z . Hence $Tz = y + \alpha z$ for some $y \in M_-$ and for some complex number α . It is easily seen α does not depend on z , and we write $\alpha = \alpha_M(T)$. If $M_- = M$, we put $\alpha_M(T) = 0$. ($M_- = \bigcup \{N \in \mathcal{F} : N \subset M\}$). See [7].)

LEMMA 1. Let $T \in \mathcal{K}(X)$ and suppose $\mathcal{F}$ is a maximal chain of subspaces of X , all invariant for T . Then $r(T) = \sup\{|\alpha_M(T)| : M \in \mathcal{F}\}$.

PROOF. Ringrose [11] proves that $\sigma(T) \setminus \{0\} = \{\alpha_M(T) : M \in \mathcal{F}\} \setminus \{0\}$, from which the above result is immediate.

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THEOREM 1. *Let A be a closed algebra of compact operators on a Banach space X . Then the following are equivalent.*

- (i) A is triangularizable.
- (ii) $r(S + T) \leq r(S) + r(T)$ ($S, T \in A$).
- (iii) $A/\text{rad } A$ is commutative.

PROOF. The equivalence of (ii) and (iii) holds for any Banach algebra and is due to Aupetit [1, p. 48].

To show (i) implies (ii), suppose A is triangularizable, and let $\mathcal{F}$ be a maximal chain of subspaces of X , all of which are invariant for A . If $S, T \in A$ and $M \in \mathcal{F}$ then it is easily seen that $\alpha_M(S + T) = \alpha_M(S) + \alpha_M(T)$. Thus

$$\begin{aligned} r(S + T) &= \sup_{M \in \mathcal{F}} |\alpha_M(S + T)| \quad (\text{by Lemma 1}) \\ &\leq \sup_{M \in \mathcal{F}} |\alpha_M(S)| + \sup_{M \in \mathcal{F}} |\alpha_M(T)| = r(S) + r(T). \end{aligned}$$

Finally to show (iii) implies (i), suppose $A/\text{rad } A$ is commutative. Then for all S, T, U in A , $S(TU - UT)$ is quasinilpotent. Thus $P(T, U)[T, U]$ is quasinilpotent for every polynomial $P(X_1, X_2)$ in two noncommuting variables. Hence by Corollary 3.5 of [7], A is triangularizable.

COROLLARY. *Every closed radical subalgebra of $\mathcal{K}(X)$ is triangularizable.*

COROLLARY [7]. *Every closed commutative subalgebra of $\mathcal{K}(X)$ is triangularizable.*

COROLLARY [7]. *If $S, T \in \mathcal{K}(X)$ and $[S, T]$ commutes with S and T , then the closed algebra A generated by S and T is triangularizable.*

PROOF. By a theorem of Kleinecke and Shirokov (see [5, p. 129]), $S[S, T] = [S, T]S$ implies $r([S, T]) = 0$. Also, if $U \in A$, then $[S, T]$ and U commute, so $r(U[S, T]) \leq r(U)r([S, T]) = 0$. Thus $[S, T] \in \text{rad } A$, and so $A/\text{rad } A$ is commutative.

REMARKS. (i) We will now give an example of a Banach space X and a closed subalgebra A of $\mathcal{B}(X)$ which is triangularizable, but for which $A/\text{rad } A$ is not commutative. This is a partial answer to questions in Remark 5.2 of [7].

Let X be the separable Hilbert space l^2 , with the canonical basis $(e_n)_{n=1}^\infty$. Let U be the unilateral shift, and W the shift with weights $(-1)^n$, so $Ue_n = e_{n+1}$ and $We_n = (-1)^n e_{n+1}$. Let A be the closed subalgebra of $\mathcal{B}(X)$ generated by U and W . A is clearly triangularizable (take $\mathcal{F} = \{M_n: n = 0, 1, 2, \dots, \infty\}$ where $M_0 = 0$, $M_\infty = X$ and M_n is the closed span of $\{e_n, e_{n+1}, \dots\}$ ($n = 1, 2, 3, \dots$)). Now $[U, W]e_n = 2(-1)^n e_{n+2}$, so $[U, W]^n e_1 = (-2)^n e_{2n-1}$ ($n = 1, 2, \dots$). Thus $\|[U, W]^n\| > 2^n$, implying $r([U, W]) > 2$. Therefore $[U, W] \notin \text{rad } A$, and so $A/\text{rad } A$ is not commutative.

(ii) The example of Enflo [4] of a Banach space X of infinite dimension and an operator T on X having only trivial invariant subspaces would imply that commutative algebras are not necessarily triangularizable (consider the closed subalgebra of $\mathcal{B}(X)$ generated by T above).

REFERENCES

1. B. Aupetit, *Propriétés spectrales des algèbres de Banach*, Lecture Notes in Math., Springer-Verlag, Berlin and New York, 1979.
2. M.-D. Choi, C. Laurie and H. Radjavi, *On commutators and invariant subspaces*, Linear and Multilinear Algebra **9** (1981), 329–340.
3. H. Dowson, *Spectral theory of linear operators*, Academic Press, London, 1978.
4. P. Enflo, *On the invariant subspace problem in Banach spaces*, Séminaire Maurey-Schwartz (1975–1976), Espaces L^p Applications Radonifiantes et Géométrie des Espaces de Banach, Exp. No. 14–15, Centre Math., Ecole Polytech., Palaiseau, 1976.
5. P. R. Halmos, *A Hilbert space problem book*, Van Nostrand, Princeton, N.J., 1967.
6. T. J. Laffey, *Simultaneous triangularization of matrices—low rank cases and the nonderogatory case*, Linear and Multilinear Algebra **6** (1978), 269–305.
7. C. Laurie, E. Nordgren, H. Radjavi and P. Rosenthal, *On triangularization of algebras of operators*, preprint.
8. V. Lomonosov, *Invariant subspaces for operators commuting with compact operators*, Funkcional. Anal. i Priložen. **7** (1973), 55–56 = Functional Anal. Appl. **7** (1973), 213–214.
9. N. H. McCoy, *On quasi-commutative matrices*, Trans. Amer. Math. Soc. **36** (1934), 327–340.
10. ———, *On the characteristic roots of matrix polynomials*, Bull. Amer. Math. Soc. **42** (1963), 592–600.
11. J. R. Ringrose, *Super-diagonal forms for compact linear operators*, Proc. London Math. Soc. (3) **12** (1962), 367–384.

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