A UNIQUENESS THEOREM FOR SUPERHARMONIC FUNCTIONS IN $\mathbb{R}^n$

J. L. SCHIFF

ABSTRACT. Let $s(x)$ be a nonnegative superharmonic function defined on the n -ball $B^n(y; r)$ in $\mathbb{R}^n$, $n > 3$. If $s(x)$ tends to zero "too rapidly" as x tends to a single point ξ on the boundary of $B^n(y; r)$, then we prove that $s \equiv 0$. The same result can then be extended to domains $D \subseteq \mathbb{R}^n$, whose boundary ∂D is locally C^1 at $\xi \in \partial D$. These results generalize some earlier work of the author and U. Kuran for $n = 2$.

1. It is quite possible for a nonnegative superharmonic function $s \not\equiv 0$ defined on a domain $D \subseteq \mathbb{R}^n$, to vanish on ∂D ; in fact, the Green's function for D (if it exists) is just such a function which vanishes everywhere on ∂D , except for possibly a polar subset of ∂D . However, in this paper, we demonstrate that $s(x)$ cannot tend to zero "too rapidly" as x approaches a point on ∂D , unless $s \equiv 0$. Similar considerations in other contexts may be found in BreLOT [1, Appendix, §26d], Kuran [2], Kuran and Schiff [3], and Naïm [4].

2. We first consider the case of the open n -ball, $B^n = B^n(y; r)$, centered at y of radius $r > 0$. The proof is a generalization of that given in [3] for $n = 2$.

THEOREM 1. *Let s be a nonnegative superharmonic function on $B^n(y; r)$, $n > 3$, such that*

$$\liminf_{\substack{x \rightarrow \xi \\ x \in B^n}} \frac{s(x)}{\text{dist}(x, \partial B^n)} = 0$$

for any $\xi \in \partial B^n$. Then $s \equiv 0$ in B^n .

PROOF. Without loss of generality we may assume $B^n = B^n(0; 1)$. By the Riesz decomposition

$$s(x) = G^\nu(x) + h(x),$$

where $G^\nu(x) = \int_{B^n} G(x, z) d\nu(z)$, $G(x, z)$ is the Green's function, ν a positive measure, and $h \geq 0$ is the greatest harmonic minorant of s on B^n . Thus

$$\liminf_{\substack{x \rightarrow \xi \\ x \in B^n}} \frac{h(x)}{1 - \|x\|} = 0.$$

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As h has the Herglotz representation,

$$h(x) = \frac{1}{\sigma_n} \int_{\partial B^n} \frac{1 - \|x\|^2}{\|x - z\|^n} d\mu(z),$$

for some positive measure μ on ∂B^n , we obtain

$$\frac{h(x)}{1 - \|x\|} > \frac{1}{2^n \sigma_n} \int_{\partial B^n} d\mu(z)$$

for all $x \in B^n$. Taking $\liminf_{\substack{x \rightarrow \xi, x \in B^n \\ x_i \in B^n}}$ yields $\int_{\partial B^n} d\mu(z) = 0$, implying $\mu \equiv 0$ and hence $h \equiv 0$ in B^n .

Likewise, for sequences $\{x_i\}$ in B^n , $x_i \rightarrow \xi$, we have by Fatou's lemma

$$\begin{aligned} 0 &= \liminf_{\substack{x_i \rightarrow \xi \\ x_i \in B^n}} \frac{G'(x_i)}{1 - \|x_i\|} = \liminf_{\substack{x_i \rightarrow \xi \\ x_i \in B^n}} \int_{B^n} \frac{G(x_i, z)}{1 - \|x_i\|} d\nu(z) \\ &> \int_{B^n} \liminf_{\substack{x_i \rightarrow \xi \\ x_i \in B^n}} \frac{G(x_i, z)}{1 - \|x_i\|} d\nu(z) > 0. \end{aligned}$$

Now, the Green's function is given by

$$G(x, z) = \frac{1}{\|z - x\|^{n-2}} - \frac{1}{\|x\|^{n-2}} \frac{1}{\|z - x^*\|^{n-2}},$$

where $x^* = x/\|x\|^2$. Setting $a = \|x\| \|z - x^*\|$, $b = \|z - x\|$, and $n - 2 = p$, yields

$$G(x, z) = \frac{a^p - b^p}{a^p b^p} = \frac{(a - b)F(a, b)}{a^p b^p},$$

where $F(a, b) = a^{p-1} + a^{p-2}b + \dots + a \cdot b^{p-2} + b^{p-1} > 0$. Thus

$$G(x, z) = \frac{(a^2 - b^2)F(a, b)}{a^p b^p (a + b)} = \frac{(\|x\|^2 \|z - x^*\|^2 - \|z - x\|^2)F(a, b)}{a^p b^p (a + b)}.$$

Hence

$$\begin{aligned} \frac{G(x, z)}{1 - \|x\|} &= \frac{(1 - \|x\|^2)(1 - \|z\|^2)F(a, b)}{(1 - \|x\|)a^p b^p (a + b)} \\ &= \frac{(1 + \|x\|)(1 - \|z\|^2)F(a, b)}{a^p b^p (a + b)}. \end{aligned}$$

We conclude that

$$\begin{aligned} f(z) &= \liminf_{\substack{x_i \rightarrow \xi \\ x_i \in B^n}} \frac{G(x_i, z)}{1 - \|x_i\|} = \frac{2(1 - \|z\|^2)F(\|z - \xi\|, \|z - \xi\|)}{\|z - \xi\|^{2n-4}(2\|z - \xi\|)} \\ &= \frac{(1 - \|z\|^2)(n - 2)\|z - \xi\|^{n-3}}{\|z - \xi\|^{2n-3}} = \frac{(n - 2)(1 - \|z\|^2)}{\|z - \xi\|^n} > 0, \end{aligned}$$

for all $z \in B^n$. As $\int_{B^n} f(z) d\nu(z) = 0$, we obtain $\nu \equiv 0$, and hence $s \equiv 0$ in B^n , as desired.

We next consider the case of a domain D in $\mathbb{R}^n$, $n > 2$, where ξ is some point on the finite boundary, ∂D , of D .

THEOREM 2. Let $D \subseteq \mathbb{R}^n$, $n \geq 2$, be such that in a neighborhood of $\xi \in \partial D$, ∂D is a C^1 -manifold, and let s be a nonnegative superharmonic function on D such that

$$\liminf_{\substack{x \rightarrow \xi \\ x \in \Delta}} \frac{s(x)}{\text{dist}(x, \partial D)} = 0,$$

where Δ is some Stolz domain at ξ . Then $s \equiv 0$ in D .

PROOF. Let $B \subseteq D$ be a sufficiently small ball such that B and ∂D share a common tangent hyperplane at ξ , and let $\Delta = \Delta(\alpha; \xi)$ be the Stolz domain of the hypothesis, which makes an angle 2α at ξ , $0 < \alpha < \pi/2$. For any $x \in \Delta \cap B \cap N_\xi$, where N_ξ is a sufficiently small neighborhood centered about ξ ,

$$\text{dist}(x, \partial D) < \text{dist}(x, \xi) < K_\alpha \cdot \text{dist}(x, \partial B),$$

where K_α is a positive constant. Thus

$$\begin{aligned} 0 &\leq \liminf_{\substack{x \rightarrow \xi \\ x \in B}} \frac{s(x)}{K_\alpha \cdot \text{dist}(x, \partial B)} \leq \liminf_{\substack{x \rightarrow \xi \\ x \in \Delta \cap B \cap N_\xi}} \frac{s(x)}{K_\alpha \cdot \text{dist}(x, \partial B)} \\ &\leq \liminf_{\substack{x \rightarrow \xi \\ x \in \Delta \cap B \cap N_\xi}} \frac{s(x)}{\text{dist}(x, \partial D)} = \liminf_{\substack{x \rightarrow \xi \\ x \in \Delta}} \frac{s(x)}{\text{dist}(x, \partial D)} = 0. \end{aligned}$$

By Theorem 1, for $n \geq 3$, and by Proposition 1 of [3], for $n = 2$, we conclude that $s \equiv 0$ in B . By the minimum principle for superharmonic functions, $s \equiv 0$ in D , which concludes the proof.

In [3], for $n = 2$, an analog of this result was established under the more restrictive boundary condition that ∂D be Dini-smooth in a neighborhood of ξ , but under the more general condition that $x \in D$ could tend to ξ via any approach in D . That version, however, requires the Riemann mapping theorem which is no longer at our disposal for $n \geq 3$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF AUCKLAND, AUCKLAND, NEW ZEALAND