A CONVERSE TO VON NEUMANN'S INEQUALITY

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ABSTRACT. The Pick-Nevanlinna theorem is used to show that if f_0 is holomorphic on an open subset G of the unit disk D and $\|f_0(T)\| < 1$ for every contraction operator T on a Hilbert space whose spectrum is contained in G , then $f_0 = f|_G$ where f is holomorphic and bounded by 1 on D .

If T is a contraction operator on a Hilbert space $\mathcal{H}$ and $p(z)$ is any polynomial such that $|p(z)| \leq 1$ on the unit disk $D = \{z: |z| < 1\}$, then

$$(1) \qquad \|p(T)\| \leq 1.$$

This is von Neumann's inequality; see [3, §4.3 and 5, p. 32]. More generally,

$$\|f(T)\| \leq 1$$

for any holomorphic function $f(z)$ such that $|f(z)| \leq 1$ on D , provided that $f(T)$ is defined by some standard functional calculus. For example, if $f(z)$ is holomorphic on an open set containing the spectrum of T , then $f(T)$ is defined by the Riesz-Dunford calculus as

$$f(T) = -\frac{1}{2\pi i} \int_{\gamma} f(\zeta)(T - \zeta I)^{-1} d\zeta$$

for a suitable contour γ . See [4, §151].

Foiaş [1] has shown that if X is a complex Banach space and (1) holds for every contraction operator T on X and every polynomial $p(z)$ such that $|p(z)| \leq 1$ on D , then X is a Hilbert space. We complement this theorem of Foiaş with a different kind of converse result.

THEOREM. Assume that f_0 is holomorphic on an open subset G of D and that

$$(2) \qquad \|f_0(T)\| < 1$$

for every contraction operator T on a Hilbert space $\mathcal{H}$ whose spectrum is contained in G . Then $f_0 = f|_G$ where f is holomorphic and bounded by 1 on D .

Under the assumptions of the theorem $f_0(T)$ is defined by the Riesz-Dunford functional calculus.

PROOF. The shift operator on $H^2(D)$ is defined by

$$S: u(z) \rightarrow zu(z), \quad u \in H^2(D).$$

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Its adjoint is given by

$$S^*: u(z) \rightarrow [u(z) - u(0)]/z, \quad u \in H^2(D).$$

Thus for each $w \in D$,

$$(3) \quad S^*: 1/(1 - wz) \rightarrow w/(1 - wz).$$

Let $w_1, \dots, w_n$ be distinct points in G . Let $\mathcal{H}$ be the span of the functions $\{1/(1 - w_j z)\}_{j=1}^n$ in $H^2(D)$, and let

$$T = S^*|_{\mathcal{H}}.$$

We view $\mathcal{H}$ as a Hilbert space in the inner product of $H^2(D)$. By (3), $S^*\mathcal{H} \subseteq \mathcal{H}$, and so T maps $\mathcal{H}$ into itself. Since S is isometric, S^* is a contraction operator on $H^2(D)$, and hence T is a contraction operator on $\mathcal{H}$. If ζ is a complex number which is not one of the numbers $w_1, \dots, w_n$, then $T - \zeta I$ is invertible and

$$(T - \zeta I)^{-1}: \sum_{j=1}^n \frac{c_j}{1 - w_j z} \rightarrow \sum_{j=1}^n \frac{c_j}{(w_j - \zeta)(1 - w_j z)}$$

for any numbers $c_1, \dots, c_n$. By (3) the numbers $w_1, \dots, w_n$ are all eigenvalues for T , so $\sigma(T) = \{w_1, \dots, w_n\} \subset G$.

Let $c_1, \dots, c_n$ be arbitrary numbers. Choose pairwise disjoint circles $\gamma_1, \dots, \gamma_n$ such that for each $j = 1, \dots, n$, γ_j is centered at w_j and contains no other point of the spectrum of T in its interior. Then

$$\begin{aligned} f_0(T) \left\{ \sum_{j=1}^n \frac{c_j}{1 - w_j z} \right\} &= - \sum_{k=1}^n \frac{1}{2\pi i} \int_{\gamma_k} f_0(\zeta) (T - \zeta I)^{-1} \left\{ \sum_{j=1}^n \frac{c_j}{1 - w_j z} \right\} d\zeta \\ &= - \sum_{k=1}^n \frac{1}{2\pi i} \int_{\gamma_k} f_0(\zeta) \sum_{j=1}^n \frac{c_j}{(w_j - \zeta)(1 - w_j z)} d\zeta \\ &= \sum_{k=1}^n \frac{f_0(w_k) c_k}{1 - w_k z} \end{aligned}$$

by Cauchy's theorem. By (2),

$$\left\| \sum_{j=1}^n \frac{f_0(w_j) c_j}{1 - w_j z} \right\|^2 \leq \left\| \sum_{j=1}^n \frac{c_j}{1 - w_j z} \right\|^2.$$

Since

$$\langle 1/(1 - az), 1/(1 - bz) \rangle = 1/(1 - a\bar{b})$$

for any numbers a and b , $|a| < 1$, $|b| < 1$, a short calculation yields

$$\sum_{j=1}^n \frac{1 - f_0(w_j) \overline{f_0(w_k)}}{1 - w_j \bar{w}_k} c_j \bar{c}_k \geq 0.$$

By the Pick-Nevanlinna theorem [2, p. 69], $f_0 = f|_G$ where f is holomorphic and bounded by 1 on D .

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