ON COMMUTING FAMILIES OF NONEXPANSIVE OPERATORS

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ABSTRACT. For each n , $1 < n < \infty$, there exist a weakly compact, convex subset W of L_1 and a family $\{T_i\}_{i=1}^n$ ($\{T_i\}_{i=1}^\infty$ in case $n = \infty$) of nonexpansive operators of W into W such that there is no common fixed point for the whole family but any (finite) strict subfamily admits a common fixed point.

Recently, Alspach [1] found an example of a weakly compact, convex subset of $L_1(0, 1)$ and a nonexpansive operator on it with no fixed points, thus solving a long-standing problem. Using the basic idea of Alspach we produce in this note a class of such examples which are somewhat simpler than Alspach's and give more variety (cf. Lemma 1 and its corollary). We use the construction to also prove the main results of this note, Theorems 1 and 2 below.

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THEOREM 1. *For every $n = 1, 2, \dots$ there exist a weakly compact, convex subset W of L_1 and a family $T_1, T_2, \dots, T_{n+1}$ of $n + 1$ commuting, nonexpansive operators of W into itself such that any n of them have a common fixed point but $T_1 \circ T_2 \circ \dots \circ T_{n+1}$ does not have a fixed point. In particular, $T_1, T_2, \dots, T_{n+1}$ do not have a common fixed point.*

THEOREM 2. *There exist a weakly compact, convex subset W of L_1 and a sequence $T_1, T_2, \dots$ of commuting nonexpansive operators of W into itself such that any finite number of them have a common fixed point but there is no common fixed point for the entire sequence.*

DEFINITION. Let $(S, \mathcal{F}, \mu)$ be a measure space and let $\tau^{-1}: S \rightarrow [0, 1] \times S$ be a measure preserving transformation. We define a (nonlinear) operator T_τ on $\{f \in L_1(S); 0 \leq f \leq 1\}$ by

$$T_\tau f(s) = \chi_{\tau(\{(t,s); 0 < t < f(s)\})}.$$

It is easily checked that T_τ is a (nonlinear) isometry. In particular, $\int T_\tau f d\mu = \int f d\mu$ for all $f \in L_1(S)$, $0 \leq f \leq 1$.

For $0 < \alpha \leq \beta < 1$ let

$$W_{\alpha,\beta} = \left\{ f \in L_1(S); 0 \leq f \leq 1, \alpha \leq \int f \leq \beta \right\}.$$

$W_{\alpha,\beta}$ is a weakly compact, convex subset of $L_1(S)$ and $T_\tau(W_{\alpha,\beta}) \subseteq W_{\alpha,\beta}$.

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LEMMA 1. Let $0 < \alpha < \beta < 1$. T_τ has a fixed point in $W_{\alpha,\beta}$ if and only if there exists a set $A \in \mathcal{F}$ with $\alpha < \mu(A) < \beta$ and $\tau([0, 1] \times A) = A$ a.e. If there exists such a set A , χ_A is a fixed point.

We leave the simple proof to the reader.

COROLLARY (D. ALSPACH). There exist a weakly compact, convex subset W of L_1 and a nonexpansive operator on W with no fixed point.

PROOF. Let $W = W_{1/2, 1/2}$, $S = [0, 1]^{\aleph_0}$ with the product Lebesgue measure and let τ be defined by $\tau(t, (s_1, s_2, \dots)) = (t, s_1, s_2, \dots)$; then $\tau([0, 1] \times A) = A$ a.e. if and only if $\mu(A) = 0$ or $\mu(A) = 1$ so T_τ does not have a fixed point.

(Another example: $S = [0, 1]$ with Lebesgue measure

$$\tau\left(\sum_{n=1}^{\infty} \varepsilon_n 2^{-n}, \sum_{n=1}^{\infty} \delta_n 2^{-n}\right) = \frac{\varepsilon_1}{2} + \frac{\delta_1}{4} + \frac{\varepsilon_2}{8} + \frac{\delta_2}{16} + \dots,$$

$\varepsilon_n, \delta_n = 0, 1, n = 1, 2, \dots$) $\square$

Let $P_S: [0, 1] \times S \rightarrow S$ be defined by $P_S(t, s) = s$. Given τ, σ such that τ^{-1}, σ^{-1} are measure preserving transformations of S onto $[0, 1] \times S$, we define $\pi = \pi(\tau, \sigma)$ by

$$\pi^{-1} = \sigma^{-1} P_S \tau^{-1}.$$

It is easily checked that π^{-1} is a measure preserving transformation, considered as a map from $(S, \mathcal{B}, \mu)$ to $[0, 1] \times (S, \mathcal{F}, \mu)$ where $\mathcal{B} = \{\tau([0, 1] \times B); B \in \mathcal{F}\}$.

LEMMA 2. With the notations as above, $T_\tau \circ T_\sigma = T_\pi$.

PROOF.

$$\begin{aligned} T_\tau \circ T_\sigma f &= T_\tau \chi_{\sigma\{(t,s); 0 < t < f(s)\}} \\ &= \chi_{\tau\{[0,1] \times \sigma\{(t,s); 0 < t < f(s)\}\}} \\ &= \chi_{\pi\{(t,s); 0 < t < f(s)\}} = T_\pi f. \quad \square \end{aligned}$$

LEMMA 3. T_τ and T_σ commute as operators on $\{f; 0 \leq f \leq 1\}$ if and only if $\tau([0, 1] \times \sigma(B)) = \sigma([0, 1] \times \tau(B))$ a.e. for all $B \subseteq [0, 1] \times S$, B measurable.

PROOF. The if part follows from the proof of the previous lemma. For the only if part notice that if T_τ and T_σ commute then taking $f = \alpha \cdot \chi_A$ in the second line of the proof of Lemma 2, we get $\tau\{[0, 1] \times \sigma(B)\} = \sigma\{[0, 1] \times \tau(B)\}$ for all B of the form $[0, \alpha] \times A$. From this one easily gets the same equality for all measurable B .

$\square$

From now on we assume $S = [0, 1]^{\aleph_0}$.

Given a one-to-one measure preserving transformation $\rho: [0, 1] \rightarrow [0, 1]$, we define $\tau = \tau(\rho): [0, 1] \times S \rightarrow S$ by

$$\tau(t, (s_1, s_2, \dots)) = (\rho(s_1), t, s_2, s_3, \dots),$$

and given $k = 1, 2, \dots$ we define $\psi^{-1}: S \rightarrow S$ by

$$\psi(s_1, s_2, \dots) = \rho(s_1) \times [0, 1]^k \times (s_2, s_3, \dots)$$

(i.e. $\psi^{-1}(s_1, s_2, \dots) = (\rho^{-1}(s_1), s_{k+2}, s_{k+3}, \dots)$).

LEMMA 4. (1) Let $A \subseteq [0, 1]$ be measurable; then $\rho(A) = A$ a.e. if and only if $\tau([0, 1] \times A \times [0, 1]^{\aleph_0}) = A \times [0, 1]^{\aleph_0}$ a.e.

(2) If ρ is ergodic (i.e. $\rho(A) = A$ a.e. implies $\mu(A) = 0$ or $\mu(A) = 1$) then ψ is ergodic.

(3) If ρ_1, ρ_2 commute then $T_{\tau(\rho_1)}, T_{\tau(\rho_2)}$ commute.

PROOF. (1) $\tau([0, 1] \times A \times [0, 1]^{\aleph_0}) = \rho(A) \times [0, 1]^{\aleph_0}$ and $\rho(A) \times [0, 1]^{\aleph_0} = A \times [0, 1]^{\aleph_0}$ a.e. if and only if $\rho(A) = A$ a.e.

(2) In the language of ergodic theory, ψ is a one-to-many transformation on a product space which is a product of an ergodic transformation with a (one-to-many) mixing transformation. It is a known fact that such a transformation is ergodic. Since books on ergodic theory usually deal only with one-to-one transformations, we reproduce the proof of this fact in our special case.

Since ψ is a measure preserving transformation, the operator T , defined on $L_2(S)$ by $Tf(s) = f(\psi^{-1}(s))$, is an isometry. If $f(s) = f_1(s_1) \cdot f_2(s_2) \cdot \dots \cdot f_l(s_l)$ and $g(s) = g_1(s_1) \cdot g_2(s_2) \cdot \dots \cdot g_l(s_l)$, then

$$T^i f = f_1(\rho^{-i}(s_1)) \cdot f_2(s_{ik+2}) \cdot f_3(s_{ik+3}) \cdot \dots \cdot f_l(s_{ik+l})$$

and

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} (T^i f, g) \\ = \left(\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} f_1 \circ \rho^{-i}, g_1 \right) \cdot \int f_2 \cdot \int \bar{g}_2 \cdot \int f_3 \cdot \int \bar{g}_3 \cdot \dots \cdot \int f_l \cdot \int \bar{g}_l \\ = \int f_1 \cdot \int \bar{g}_1 \cdot \int f_2 \cdot \int \bar{g}_2 \cdot \dots \cdot \int f_l \cdot \int \bar{g}_l = \int f \cdot \int \bar{g}. \end{aligned}$$

The limit in the second term is in L_2 ; its existence and equality to $\int f_1$ follow from the simplest form of the ergodic theorem (cf. [2]). It follows that for any $f, g \in L_2$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} (T^i f, g) = \int f \cdot \int \bar{g}.$$

In particular, if A is such that $\psi(A) = A$ then $T^i \chi_A = \chi_A$ for all i and

$$(\chi_A, g) = \mu(A) \cdot \int \bar{g} \quad \text{for all } g \in L_2.$$

For $g = \chi_A$ we get $\mu(A) = \mu(A)^2$ so $\mu(A) = 0$ or $\mu(A) = 1$.

(3) is easily checked using Lemma 3. $\square$

PROOF OF THEOREM 2. $W = W_{1/2, 1/2}$. Let $\{r_i\}_{i=1}^{\infty}$ be an enumeration of the rationals in $[0, 1]$, let ρ_i be defined by $\rho_i(t) = t + r_i \pmod{1}$ and let $T_i = T_{\tau(\rho_i)}$. By Lemma 4.3 $\{T_i\}_{i=1}^{\infty}$ commute. Given n let m be a common denominator for $r_1, \dots, r_n$ and let $A = \bigcup_{k=0}^{m-1} (k/m, k/m + 1/2m)$; then $\mu(A) = \frac{1}{2}$, and $\rho_i(A) = A$ for $i = 1, \dots, n$. By Lemmas 1 and 4.1 χ_A is a common fixed point for $T_1, \dots, T_n$. If χ_A is a fixed point for all the T_i 's, then $\tau(\rho_i)([0, 1] \times A) = A$ for $i = 1, 2, \dots$. It follows easily that $\tau(\rho)([0, 1] \times A) = A$ for any ρ of the form $\rho(t) = t + \alpha \pmod{1}$, $\alpha \in [0, 1]$. However, if α is irrational, ρ is ergodic in contradiction to Lemma 4.2 (with $k = 1$). $\square$

Before starting the proof of Theorem 1 let us illustrate the idea by looking at the case $n = 1$. Divide the interval $[0, 1]$ into six successive intervals $I_1, I_2, \dots, I_6$ each of length $\frac{1}{6}$. Let $\varepsilon: [0, 1] \rightarrow [0, 1]$ be a measure preserving transformation such that $\varepsilon(I_i) = I_i$, $i = 1, \dots, 6$, and ε^{12} is ergodic relative to each of the I_i , i.e. I_i , $i = 1, \dots, 6$, and their finite unions are the only nontrivial ε^{12} -invariant sets. Let ρ_1 be the transformation which carries I_i linearly onto $I_{i+2 \pmod 6}$ followed by ε , and let ρ_2 be the transformation which switches I_1 with I_2 , I_3 with I_4 , I_5 with I_6 followed by ε . Clearly, $I_1 \cup I_3 \cup I_5$ is invariant under ρ_1 while $I_1 \cup I_2$ is invariant under ρ_2 . The composition $\rho_1 \circ \rho_2 = \rho_2 \circ \rho_1$ is the transformation which carries linearly $I_1 \rightarrow I_4 \rightarrow I_5 \rightarrow I_2 \rightarrow I_3 \rightarrow I_6 \rightarrow I_1$ followed by ε^{12} and it is easily seen to be ergodic. If we now define $T_i = T_{\tau(\rho_i)}$, $i = 1, 2$, we get the desired result for $n = 1$.

PROOF OF THEOREM 1. Fix n , let $p_1, \dots, p_{n+1}$ be the first $n+1$ prime numbers ($p_1 = 2$) and let $p = p_1 p_2 \cdots p_{n+1}$. Let $\varepsilon: [0, p^{-1}) \rightarrow [0, p^{-1})$ be a one-to-one measure preserving transformation with ε^k ergodic for all $k = 1, 2, \dots$. For $i = 1, 2, \dots, n+1$ define a measure preserving transformation ρ_i of $[0, 1)$ onto itself by

$$\rho_i(x) = (\varepsilon(x - kp^{-1}) + kp^{-1} + p_i^{-1}) \pmod{1} \quad \text{if } kp^{-1} \leq x < (k+1)p^{-1}.$$

Let $\bar{\tau}_i = \tau(\rho_i)$ and $\bar{T}_i = T_{\bar{\tau}_i}$.

Given any subset of $\{\bar{T}_i\}_{i=1}^{n+1}$ of cardinality n , say $\{\bar{T}_i\}_{i=1, i \neq j}^{n+1}$, let $A = \bigcup_{k=0}^{pp_j^{-1}-1} [kp_j p^{-1}, kp_j p^{-1} + p^{-1})$; then $\mu(A) = p_j^{-1}$ and $\rho_i(A) = A$ for all $i \neq j$. It follows from Lemmas 1 and 4.1 that $\chi_{A \times [0, 1]^{n_0}}$ is a fixed point of $\bar{T}_i$ for all $i \neq j$.

By the Chinese Remainder Theorem there are positive integers $a_1, a_2, \dots, a_{n+1}$ such that $p^{-1} = \sum_{i=1}^{n+1} a_i p_i^{-1} \pmod{1}$. Put $T_i = (\bar{T}_i)^{a_i}$, $i = 1, \dots, n+1$. We claim that $\{T_i\}_{i=1}^{n+1}$ and $W = W_{\alpha, \beta}$, where $\alpha = \min_{1 \leq i \leq n+1} p_i^{-1}$, $\beta = \frac{1}{2}$, satisfy the conclusion of Theorem 1.

From the first part of the proof it is clear that any n of $T_1, \dots, T_{n+1}$ have a common fixed point. Since $\rho_1, \dots, \rho_{n+1}$ commute, $\bar{T}_1, \dots, \bar{T}_{n+1}$ and, thus, $T_1, \dots, T_{n+1}$ commute. To show that $T = T_1 \circ \dots \circ T_{n+1}$ does not have a fixed point notice that, by Lemma 2, $T = T_\psi$ where

$$\psi(t, (s_1, s_2, \dots)) = (\rho_1^{a_1} \circ \dots \circ \rho_{n+1}^{a_{n+1}})(s_1) \times [0, 1]^a \times (t, s_2, s_3, \dots),$$

$a = a_1 + a_2 + \dots + a_{n+1}$. By Lemmas 1 and 4.2 it is enough to show that $\rho = \rho_1^{a_1} \circ \dots \circ \rho_{n+1}^{a_{n+1}}$ is ergodic, but $\rho(x) = (\varepsilon^a(x - kp^{-1}) + (k+1)p^{-1}) \pmod{1}$ if $kp^{-1} \leq x < (k+1)p^{-1}$, and the ergodicity of ρ follows easily from that of ε^a .

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