A NOTE ON IDEALS IN THE DISC ALGEBRA

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ABSTRACT. We offer an elementary theorem on ideals in the disc algebra $A(\mathbf{D})$, which by way of a corollary, one, identifies the maximal ideals of $A(\mathbf{D})$, and two, provides a proof, which avoids the axiom of choice, that every proper ideal in $A(\mathbf{D})$ is contained in a maximal ideal.

1. Let A be a ring of functions in the closed unit disc $\overline{\mathbf{D}}$, each of which is continuous in $\overline{\mathbf{D}}$ and holomorphic in $\mathbf{D}$. (Thus $A \leq A(\mathbf{D})$.) Suppose that

(a) A is dense in the disc algebra $A(\mathbf{D})$, i.e. if $f \in A(\mathbf{D})$ and $\varepsilon > 0$, then there is a g in A with $|f - g| < \varepsilon$ in $\mathbf{D}$.

Then we have the following.

THEOREM. Let Q be an ideal in A , $Q \neq 0$, and let

$$(1) \quad X \subset \bigcup_{f \in Q} \{f \neq 0\}$$

where by $\{f \neq 0\}$ we mean the set of those z in $\overline{\mathbf{D}}$ for which $f(z) \neq 0$. If X is closed in $\overline{\mathbf{D}}$, or if $A = A(\mathbf{D})$ and

$$(2) \quad X \cap \mathbf{T} = \overline{X} \cap \mathbf{T}$$

where by $\overline{X}$ we mean the closure of X in $\overline{\mathbf{D}}$, then there is an f in Q with f vanishing nowhere in X .

We will come to the proof in due course. The theorem raises the following question (that we are unable to answer). Which ideals Q in A , $Q \neq 0$, have the property that if X is equal to the right side of (1), then there is an f in Q with f vanishing nowhere in X ? This holds if Q is principal. What if Q is finitely generated?

Does it hold if Q is an ideal of denominators, i.e. if

$$(3) \quad Q = Q(\gamma) = \{f: f \in A, f\gamma \in A\}$$

where γ is in the field of fractions $A_{(0)}$ of A ? If yes, this would imply (see the proof of the corollary below) that if X is any subset of $\overline{\mathbf{D}}$, $X \neq \emptyset$, then

$$(4) \quad A_X = \bigcap_{f \in X} A_f$$

where by A_X we mean the ring of fractions

$$\{g/f: g, f \in A, f \text{ vanishes nowhere in } X\},$$

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and by A_ζ , $\zeta \in \bar{\mathbf{D}}$, we mean the local ring of fractions

$$\{g/f: g, f \in A, f(\zeta) \neq 0\}.$$

If not, then one might ask for which $X \subset \bar{\mathbf{D}}$ does (4) hold. In this regard we have the following.

COROLLARY. *Let $X \subset \bar{\mathbf{D}}$, $X \neq \emptyset$. If X is closed in $\bar{\mathbf{D}}$, or if $A = A(\mathbf{D})$ and (2) holds, then (4) holds.*

PROOF. Let γ belong to the right side of (4), and put $Q =$ the right side of (3). Then (1) holds; hence there is an f in Q with f vanishing nowhere in X . This proves that $\gamma \in A_X$.

Suppose that in addition to (a) we have

(b) If $f \in A$, and f vanishes nowhere in $\bar{\mathbf{D}}$, then $1/f \in A$.

Then the following holds.

COROLLARY. *Let Q be an ideal in A , $Q \neq A$. Then there is a ζ in $\bar{\mathbf{D}}$ such that $Q \subset P_\zeta$ where*

$$P_\zeta = \{f: f \in A, f(\zeta) = 0\}.$$

PROOF. Otherwise, in the statement of the theorem we may take $X = \bar{\mathbf{D}}$.

Our proof of the theorem is very elementary; its main ingredient is that if

$$\varphi = \sum_{-k}^k \alpha_j e^{ij\theta}$$

is a trigonometric polynomial, then

$$(5) \quad e^{ik\theta} \varphi = g|_{\mathbf{T}}$$

where $g \in A(\mathbf{D})$.

By the second corollary, every maximal ideal in A is a P_ζ , $\zeta \in \bar{\mathbf{D}}$; on the other hand, if A is an algebra over $\mathbf{C}$, then every P_ζ is maximal. This identifies the maximal ideals of $A(\mathbf{D})$ in a way which is more elementary than those in [3]. To turn the second corollary around, we have an easy and elementary proof (which does not use the axiom of choice) of the fact that if $f_1, \dots, f_n$ in A do not have a common zero in $\bar{\mathbf{D}}$, then $(f_1, \dots, f_n) = (1)$. Other proofs of this (if $A = A(\mathbf{D})$) which avoid the axiom of choice are in [1] and [2].

We might point out that Theorem 1 is to some extent peculiar to the disc $\mathbf{D}$. For example, it fails for the ball algebra $A(\mathbf{B})$ in 2 variables (let $Q = P_0 = \{f: f \in A(\mathbf{B}), f(0) = 0\}$, $X = \partial\mathbf{B} = \{(z, w): |z|^2 + |w|^2 = 1\}$).

2. We now come to the proof of the theorem. To begin, let $A = A(\mathbf{D})$. Since $X \cap \mathbf{T}$ is compact, we have

$$X \cap \mathbf{T} \subset \bigcup_{j=1}^n \{f_j \neq 0\}$$

where $f_1, \dots, f_n \in Q$. Let $Y = X \cap \mathbf{T}$. WLOG we may assume that $Y \neq \emptyset$. Put

$$f = \sum_{j=1}^n \bar{f}_j f_j,$$

let

$$\alpha = \inf_Y f, \quad \beta = \inf_{\mathbf{T}} \left(1 / \sum_{j=1}^n |f_j| \right),$$

and choose trigonometric polynomials $\varphi_1, \dots, \varphi_n$ such that $|\bar{f}_j - \varphi_j| < \alpha\beta$. Then (on $\mathbf{T}$)

$$\left| f - \sum_{j=1}^n \varphi_j f_j \right| = \left| \sum_{j=1}^n (\bar{f}_j - \varphi_j) f_j \right| < \alpha,$$

hence $\sum \varphi_j f_j$ vanishes nowhere in Y . Replacing φ_j by $e^{ik\theta} \varphi_j$ (where $k \geq \deg \varphi_j$, $1 \leq j \leq n$), we obtain (cf. (5)) $g_1, \dots, g_n$ in $A(\mathbf{D})$ with $\sum g_j f_j$ vanishing nowhere in Y . Let

$$g = \sum_{j=1}^n g_j f_j.$$

Then $g \in (f_1, \dots, f_n)$. Since g vanishes nowhere in $\bar{X} \cap \mathbf{T}$, g has at most a finite number of zeros in X (counting multiplicities), say $\zeta_1, \dots, \zeta_m$. We have $\zeta_j \in \mathbf{D}$, hence

$$(6) \quad g(z) = (z - \zeta_1) \cdots (z - \zeta_m) h(z)$$

where $h \in A(\mathbf{D})$, h vanishes nowhere in X .

Let $1 \leq j \leq m$, and choose h_j in Q with $h_j(\zeta_j) \neq 0$. We have

$$h_j(z) - h_j(\zeta_j) = (z - \zeta_j) \psi_j(z)$$

where $\psi_j \in A(\mathbf{D})$, i.e.

$$(7) \quad h_j(\zeta_j) = h_j - (z - \zeta_j) \psi_j.$$

Put

$$\mu = \prod_{j=1}^m h_j(\zeta_j);$$

then (6) and (7) give

$$\mu h \in (f_1, \dots, f_n, h_1, \dots, h_m).$$

Thus $h \in Q$, which completes the proof of the theorem if $A = A(\mathbf{D})$.

If $A \neq A(\mathbf{D})$, put

$$P = \left\{ \sum_1^k f_j g_j : f_j \in A(\mathbf{D}), g_j \in Q \right\}.$$

Then P is an ideal in $A(\mathbf{D})$, with $Q \subset P$; by the foregoing there is an f in P with f vanishing nowhere in X . We have

$$f = \sum_{j=1}^n f_j g_j$$

where $f_j \in A(\mathbf{D})$, $g_j \in Q$. Let

$$\alpha = \inf_X |f|, \quad \beta = \inf_{\mathbf{D}} \left(1 / \sum_{j=1}^n |g_j| \right).$$

By the hypothesis (a) we may choose h_j in A , $1 \leq j \leq n$, with $|f_j - h_j| < \alpha\beta$. Put

$$h = \sum_{j=1}^n h_j g_j.$$

Then $h \in Q$; and

$$|f - h| = \left| \sum_{j=1}^n (f_j - h_j) g_j \right| < \alpha,$$

hence h vanishes nowhere in X .

3. In the foregoing we may replace $\bar{\mathbf{D}}$ by any bordered Riemann surface $\bar{W}$ which is compact. Our proof of the theorem (and its corollaries) would then work for any ring of functions in $A(W)$ which is dense in $A(W)$. One should replace the trigonometric polynomials on $\mathbf{T}$ by the separating selfadjoint algebra in $C(\partial W)$ consisting of the complex linear span of quotients of inner functions in $A(W)$ (cf. [4], where it is proved that the inner functions in $A(W)$ separate points in ∂W).

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