ON JORDAN REPRESENTATIONS OF UNBOUNDED OPERATOR ALGEBRAS

SUBHASH J. BHATT

ABSTRACT. Every closed Jordan $*$ -representation of an EC^* -algebra is the sum of a closed $*$ -representation and a closed $*$ -antirepresentation.

Atsushi Inoue [3] has recently initiated the study of a class of unbounded operator algebras, called EC^* -algebras. They seem to be quite useful in connection with unbounded Hilbert algebras. The purpose of this note is to prove the following theorem concerning the structure of unbounded J^* -representations of these algebras.

THEOREM. *Every closed J^* -representation π of an EC^* -algebra A is a direct sum of a closed $*$ -representation π_1 and a closed $*$ -antirepresentation π_2 .*

A mapping π of a $*$ -algebra A with identity 1 into linear operators (not necessarily bounded) all defined on a common domain $D(\pi)$ dense in a Hilbert space K is called a J^* -representation of A on K if, for all $x, y \in A$, $\alpha, \beta \in \mathbb{C}$ and $\xi, \eta \in D(\pi)$, the following hold:

1. $\pi(\alpha x + \beta y)\xi = \alpha\pi(x)\xi + \beta\pi(y)\xi$,
2. $\langle \pi(x)\xi, \eta \rangle = \langle \xi, \pi(x^*)\eta \rangle$,
3. $\pi(x)D(\pi) \subset D(\pi)$ and $\pi(xy + yx)\xi = \pi(x)\pi(y)\xi + \pi(y)\pi(x)\xi$,
4. $\pi(1) = I$.

The closure $\bar{\pi}$ of π is a J^* -representation on K defined as $\bar{\pi}(x) = \overline{\pi(x)}$ ($x \in A$), with domain $D(\bar{\pi}) = \bigcap \{D(\overline{\pi(x)}) \mid x \in A\}$ where $D(\overline{\pi(x)})$ denotes the domain of the closure $\overline{\pi(x)}$ of the operator $\pi(x)$ in K . The map π is *closed* if $\pi = \bar{\pi}$.

Our theorem is an extension of a result of Størmer [6, Theorem 3.3] on C^* -algebras. Let A be an EC^* -algebra on a dense subspace D of a Hilbert space H . Then its bounded part $\bar{A}_b = \{\bar{T} \mid T \text{ is a bounded operator in } A\}$ is a C^* -algebra. We apply Størmer's theorem to $\bar{A}_b$. Since EC^* -algebras are concrete realizations of locally convex GB^* -algebras [2, Theorem 7.11], the proof is completed by applying next a density theorem [1] viz. If A is a GB^* -algebra with unit ball B_0 , then the B^* -algebra $A(B_0)$ is sequentially dense in A . Note that $A(B_0) = \{\lambda x \mid \lambda \in \mathbb{C}, x \in B_0\}$ with the Minkowski functional $\|\cdot\|_{B_0}$ as the norm.

We shall need the following elementary lemmas.

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LEMMA A. Let A be a locally convex GB^* -algebra with unit ball B_0 . Let π be a J^* -representation of A on K . Then $x \rightarrow \pi(x)$ defines a J^* -homomorphism of the B^* -algebra $A(B_0)$ into $B(K)$, the algebra of all bounded linear operators on K .

We omit the easy proof.

LEMMA B. Let A be a locally convex GB^* -algebra. Let τ be the largest locally convex GB^* -topology on A . Let π be a J^* -representation of A on K . Let σ_π be the weak topology on $\pi(A)$ defined by the seminorms $\pi(x) \rightarrow |\langle \pi(x)\xi, \eta \rangle|$ for ξ, η in $D(\pi)$. Then $\pi: (A, \tau) \rightarrow (\pi(A), \sigma_\pi)$ is continuous.

PROOF. By the polarization identity, σ_π is also determined by the seminorms $p(\pi(x)) = |\langle \pi(x)\xi, \xi \rangle|$ for $\xi \in D(\pi)$. For any such ξ , let f on A be defined by $f(x) = \langle \pi(x)\xi, \xi \rangle$. Let $x \in A$. By [2, Proposition 5.1], $x^*x = h^2$ for some $h \geq 0$. Then $f(x^*x) = \langle \pi(h^2)\xi, \xi \rangle = \langle \pi(h)^2\xi, \xi \rangle = \|\pi(h)\xi\|^2 \geq 0$. Thus f is a positive linear functional on A which by [2, §8] is τ -continuous. Now the result follows immediately.

PROOF OF THE THEOREM. Let A be an EC^* -algebra over a domain D dense in H , a Hilbert space. Let $\bar{A} = \{\bar{T} \mid T \in A\}$. Let σ be the weak topology on A defined by the seminorms $T \rightarrow p_{\xi, \eta}(T) = |\langle T\xi, \eta \rangle|$ for $\xi, \eta \in D$. Then [2, Theorem 7.12] shows that $(\bar{A}, \sigma)$ is a locally convex GB^* -algebra with unit ball $B_0 = \{\bar{T} \in \bar{A} \mid \|T\| \leq 1\}$ so that the underlying B^* -algebra is $\bar{A}_b = \bar{A} \cap \beta(H)$. Let τ be the largest locally convex GB^* -topology on A [2, §6]. From now on, for the sake of simplicity, we omit writing the bar over the elements of A .

By Lemma A, $T \rightarrow \phi(T) = \pi(\bar{T})$ defines a J^* -representation of $\bar{A}_b$ into $\beta(K)$. Let B be the C^* -algebra in $B(K)$ generated by $\phi(\bar{A}_b)$. Hence [6, Theorem 3.3] there exist orthogonal central projections E and F in the von Neumann algebra generated by B such that $T \rightarrow \phi_1(T) = \phi(T)E$ defines a $*$ -representation of $\bar{A}_b$ on $K_1 = EK$ and $T \rightarrow \phi_2(T)F$ defines a $*$ -antirepresentation of $\bar{A}_b$ on $K_2 = FK$. Also $E + F = I$. Let $D(\pi_1) = ED(\pi)$, $D(\pi_2) = FD(\pi)$.

We show that, for each $T \in A$, $\xi, \eta \in D(\pi)$, $\langle E\pi(T)\xi, \eta \rangle = \langle E\xi, \pi(T)^*\eta \rangle$. Indeed, by the density theorem, there is a sequence T_n in A_b such that $T_n \rightarrow T$ in τ . By Lemma B,

$$\begin{aligned} \langle E\pi(T)\xi, \eta \rangle &= \langle \pi(T)\xi, E\eta \rangle = \lim_n \langle \pi(T_n)\xi, E\eta \rangle = \lim_n \langle E\pi(T_n)\xi, \eta \rangle \\ &= \lim_n \langle \pi(T_n)E\xi, \eta \rangle \quad \text{as } E \text{ is central} \\ &= \lim_n \langle E\xi, \pi(T_n)^*\eta \rangle \\ &= \lim_n \langle E\xi, \pi(T_n^*)\eta \rangle \quad \text{as } \pi(T_n^*) \subset \pi(T_n)^* \\ &= \langle E\xi, \pi(T^*)\eta \rangle \\ &\quad \text{again by Lemma B and the continuity of the involution} \\ &= \langle E\xi, \pi(T)^*\eta \rangle. \end{aligned}$$

This shows that $\eta \rightarrow \langle E\xi, \pi(T)^*\eta \rangle$ defines, for each $\xi \in D(\pi)$, a norm bounded linear functional on $D(\pi)$, and so $E\xi \in D(\pi(T)^{**}) = D(\pi(\bar{T}))$. Thus

$E\xi \in \cap\{\overline{D(\pi(T))} \mid T \in A\} = D(\bar{\pi}) = D(\pi)$. Thus $ED(\pi) \subset D(\pi)$. Similarly $FD(\pi) \subset D(\pi)$ and so $D(\pi_1) + D(\pi_2) \subset D(\pi)$, $D(\pi_1) + D(\pi_2) = D(\pi)$.

Clearly $D(\pi_1)$ is dense in K_1 and $D(\pi_2)$ is dense in K_2 . Let, for each $T \in A$, $\pi_1(T) = \pi(T)E$, $\pi_2(T) = \pi(T)F$ with domains $D(\pi_1)$ and $D(\pi_2)$ respectively. We show that π_1 and π_2 are the required maps.

Let $T \in A$. For each $n = 1, 2, \dots$ let $T_n = T(1 + \frac{1}{n}T^*T)^{-1}$. (Here the sum and the product are in the strong sense.) First we show that $T_n \rightarrow T$ in τ .

$$\begin{aligned} T - T_n &= \frac{1}{n}TT^*T(1 + \frac{1}{n}T^*T)^{-1} \\ &= \frac{1}{\sqrt{n}}(TT^*)\left(\frac{T}{\sqrt{n}}\right)\left(1 + \left(\frac{T}{\sqrt{n}}\right)^*\left(\frac{T}{\sqrt{n}}\right)\right)^{-1} \in \frac{1}{\sqrt{n}}(TT^*)B_0 \\ &\quad \text{by [5, Theorem 13.13].} \end{aligned}$$

Now by the separate continuity of multiplication in τ , given an σ -neighbourhood V , there exists an σ -neighbourhood U such that $TT^*U \subset V$. Further, as B_0 is τ -bounded, $\sqrt{r}B_0 \subset U$ for sufficiently small $r > 0$. It follows that $T - T_n \in V$ eventually. Thus $T_n \rightarrow T$ in τ .

Next we show that $\pi(A)D(\pi_1) \subset D(\pi_1)$ and $\pi(A)D(\pi_2) \subset D(\pi_2)$. Let $\xi, \eta \in D(\pi)$. By Lemma B,

$$\begin{aligned} \langle \pi(T)E\xi, \eta \rangle &= \lim_n \langle \pi(T_n)E\xi, \eta \rangle \\ &= \lim_n \langle E\pi(T_n)E\xi, \eta \rangle \quad \text{as } \pi(T_n)E\xi \in K \\ &= \lim_n \langle \pi(T_n)E\xi, E\eta \rangle \\ &= \langle \pi(T)E\xi, E\eta \rangle \quad \text{again by Lemma B} \\ &= \langle E\pi(T)E\xi, \eta \rangle \end{aligned}$$

and so $\pi(T)E\xi = E\pi(T)E\xi \in D(\pi_1)$. Hence $\pi(A)D(\pi_1) \subset D(\pi_1)$. Similarly $\pi(A)D(\pi_2) \subset D(\pi_2)$. Thus $\pi_1(T)$ and $\pi_2(T)$ are operators in K_1 and K_2 respectively.

Now given T, S in A , again by the density theorem, there exist sequences T_n and S_n in A such that $T_n \rightarrow T$, $S_n \rightarrow S$ in τ . As (A, τ) is barrelled [2, Lemma 6.2], it is hypocontinuous by [2, Lemma 6.3]. Since the multiplication in a hypocontinuous algebra is easily seen to be sequentially jointly continuous, $T_n S_n \rightarrow TS$. Then for each ξ, η in $D(\pi_1)$, repeated uses of Lemma B give

$$\begin{aligned} \langle \pi_1(T)\pi_1(S)\xi, \eta \rangle &= \lim_n \lim_k \langle \pi_1(T_n)\pi_1(S_k)\xi, \eta \rangle \\ &= \lim_n \langle \pi_1(T_n)\pi_1(S_n)\xi, \eta \rangle \\ &= \lim_n \langle \pi_1(T_n S_n)\xi, \eta \rangle \quad \text{as } \phi_1 \text{ is a representation} \\ &= \langle \pi_1(TS)\xi, \eta \rangle. \end{aligned}$$

Thus π_1 is a $*$ -representation. Similarly π_2 is a $*$ -antirepresentation.

It only remains to show that each of π_1 and π_2 is closed. Let $T \in A$ and $\xi \in \overline{D(\pi_1(T))}$, the domain of the closure in K . Then for some sequence $\{\xi_n, \pi_1(T)\xi_n\}$

in the graph of $\pi_1(T)$, $\xi_n \rightarrow \xi$ in K_1 . Then clearly $\xi \in D(\overline{\pi(T)})$ and so $\xi \in D(\bar{\pi}) = D(\pi)$. Also π_1 is closed, and similarly so is π_2 .

This completes the proof of the theorem.

Note that what is essentially required in the proof is the fact that A is a locally convex GB^* -algebra. Hence [4, Corollary 3.4] immediately gives the following.

COROLLARY. *Let D be a pure unbounded Hilbert algebra over a maximal unital Hilbert algebra D_0 . Then each closed J^* -representation of D is the direct sum of a closed $*$ -representation and a closed $*$ -antirepresentation.*

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DEPARTMENT OF MATHEMATICS, SARDAR PATEL UNIVERSITY, VALLABH VIDYANAGAR-388120, INDIA