COMPLETELY METRIZABLE SPACES OF EMBEDDINGS

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ABSTRACT. A study is made of the complete metrizability of spaces of embeddings and spaces of homeomorphisms under the compact-open topology. It is shown that the space of continuous injections from a hemicompact metric space into a complete metric space is completely metrizable.

If X is a hemicompact k -space and Y is completely metrizable, then the space $C(X, Y)$ of continuous functions from X into Y with the compact-open topology is completely metrizable (see [2] and [3]). The converse is also true if Y contains an arc. But complete metrizability is not a hereditary property, so that certain natural subspaces of $C(X, Y)$, such as the space $E(X, Y)$ of embeddings of X into Y , may not be completely metrizable whenever $C(X, Y)$ is completely metrizable. Even the space $H(X)$ of homeomorphisms from X into itself may not be completely metrizable whenever X is a hemicompact complete metric space. The following theorem shows however that the space $I(X, Y)$ of continuous injection is in fact completely metrizable assuming reasonable properties of X and Y . All function spaces will have the compact-open topology.

THEOREM. *If X is a hemicompact metric space and Y is a complete metric space, then $I(X, Y)$ is completely metrizable.*

PROOF. Let us first assume that X is compact. Since Y is a complete metric space, the supremum metric d on $C(X, Y)$ is a compatible complete metric. Therefore it suffices to show that $I(X, Y)$ is a G_δ -set in $C(X, Y)$. To this end, for each ϕ in the space of real-valued continuous functions on X , $C(X)$, and for each $\epsilon > 0$, define

$$G(\phi, \epsilon) = \{f \in C(X, Y) \mid d(\phi, \psi \circ f) < \epsilon \text{ for some } \psi \in C(Y)\}.$$

To see that each $G(\phi, \epsilon)$ is open in $C(X, Y)$, let $f \in G(\phi, \epsilon)$. Then $d(\phi, \psi \circ f) < \epsilon$ for some $\psi \in C(Y)$; let $\delta = \epsilon - d(\phi, \psi \circ f)$. Since $f(X)$ is compact, there exists a Lebesgue number λ for $\{\psi^{-1}(B(r, \delta/2)) \mid r \in \mathbb{R}\}$ as an open cover of $f(X)$, where $B(r, \delta/2)$ stands for the open ball of radius $\delta/2$ centered at r . Let $g \in B(f, \lambda)$, and let $x \in X$. Then $|\psi(f(x)) - \psi(g(x))| < \delta$, so that

$$\begin{aligned} |\phi(x) - \psi(g(x))| &\leq |\phi(x) - \psi(f(x))| + |\psi(f(x)) - \psi(g(x))| \\ &< d(\phi, \psi \circ f) + \delta = \epsilon. \end{aligned}$$

Therefore $B(f, \lambda) \subseteq G(\phi, \epsilon)$, making $G(\phi, \epsilon)$ open in $C(X, Y)$.

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Let $\{\phi_n | n \in \mathbb{N}\}$ be a countable dense subset of $C(X)$, which exists since X is a separable metric space. To see that

$$I(X, Y) = \bigcap \{G(\phi_n, 1/m) | m, n \in \mathbb{N}\},$$

let $h \in I(X, Y)$, let $\phi \in C(X)$, and let $\varepsilon > 0$. Since $h(X)$ is closed in Y , $\phi \circ h^{-1}: h(X) \rightarrow \mathbb{R}$ has a continuous extension, ψ , to all of Y . Then $\psi \circ h = \phi$, so that $h \in G(\phi, \varepsilon)$. Therefore $I(X, Y) \subseteq G(\phi, \varepsilon)$ for every $\phi \in C(X)$ and $\varepsilon > 0$. To establish containment in the other direction, let $f \in \bigcap \{G(\phi_n, 1/m) | m, n \in \mathbb{N}\}$. We must show that f is one-to-one, so let x and y be distinct elements of X . Since $\{\phi_n | n \in \mathbb{N}\}$ is dense in $C(X)$, there exists an n such that $\phi_n(x) \neq \phi_n(y)$. Now choose $m \in \mathbb{N}$ such that $2/m < |\phi_n(x) - \phi_n(y)|$. If $f(x)$ were equal to $f(y)$, then for every $\psi \in C(Y)$,

$$\begin{aligned} 2/m &< |\phi_n(x) - \phi_n(y)| \\ &\leq |\phi_n(x) - \psi(f(x))| + |\psi(f(x)) - \psi(f(y))| + |\psi(f(y)) - \phi_n(y)| \\ &= |\phi_n(x) - \psi(f(x))| + |\phi_n(y) - \psi(f(y))| \\ &\leq 2 d(\phi_n, \psi \circ f). \end{aligned}$$

But this says that $d(\phi_n, \psi \circ f) > 1/m$ for every $\psi \in C(Y)$, which contradicts f being in $G(\phi_n, 1/m)$. Therefore $f(x) \neq f(y)$, so that f is in $I(X, Y)$.

We now consider the more general case that X is hemicompact. Then we may write $X = \bigcup \{A_n | n \in \mathbb{N}\}$, where each A_n is a compact subset of X , each A_n is contained in the interior of A_{n+1} , and each compact subset of X is contained in some A_n . Then by the previous part of the proof, $\Pi\{I(A_n, Y) | n \in \mathbb{N}\}$ is completely metrizable. It will suffice to embed $I(X, Y)$ as a closed subspace of $\Pi\{I(A_n, Y) | n \in \mathbb{N}\}$.

Let ϕ be the natural projection from the disjoint sum $\Sigma\{A_n | n \in \mathbb{N}\}$ onto X . Since this is a continuous compact-covering map, the induced function

$$\phi^*: C\left(\sum_n A_n, Y\right) \rightarrow C(X, Y),$$

defined by $\phi^*(f) = f \circ \phi$, is an embedding. Now there is a natural homeomorphism α from $C(\sum_n A_n, Y)$ onto $\prod_n C(A_n, Y)$. If i is the inclusion map from $I(X, Y)$ into $C(X, Y)$, then define

$$\Phi: I(X, Y) \rightarrow \prod_n C(A_n, Y)$$

by $\Phi = \alpha \circ \phi^* \circ i$, which is an embedding. Let Z be the image of Φ . It should be clear that Z is actually contained in $\prod_n I(A_n, Y)$.

To see that Z is closed in $\prod_n I(A_n, Y)$, let $\langle f_n \rangle \in (\prod_n I(A_n, Y)) - Z$. Suppose that for every n , $f_{n+1}|_{A_n} = f_n$. Then since each $A_n \subseteq \text{int } A_{n+1}$, $\{f_n|_{\text{int } A_n} | n \in \mathbb{N}\}$ is a compatible family of continuous functions whose domains form an open cover of X . Therefore the combination function f is continuous, and will also be one-to-one. But then $f \in I(X, Y)$, while $\Phi(f) = \langle f_n \rangle$, which is a contradiction. Therefore there exists an $n \in \mathbb{N}$ and an $x \in A_n$ such that $f_{n+1}(x) \neq f_n(x)$. Let V and W be disjoint

open subsets of Y containing $f_n(x)$ and $f_{n+1}(x)$, respectively. Then if $[[x, V]]$ denotes the set of functions taking x into V ,

$$\langle f_n \rangle \in \pi^{-1}([[x, V]]) \cap \pi_{n+1}^{-1}([[x, W]]) \subseteq \left(\prod_n I(A_n, Y) \right) - Z.$$

Since the space $E(X, Y)$ of embeddings of X into Y is equal to $I(X, Y)$ when X is compact, then we have the following special case (which can be found in [4, §5.1]).

COROLLARY 1. *If X is a compact metric space and Y is a complete metric space, then $E(X, Y)$ is completely metrizable.*

Also since the space $H(X)$ of homeomorphisms from X onto itself is a closed subspace of $E(X, X)$ when X is compact, then we obtain a result about $H(X)$.

COROLLARY 2. *If X is a compact metric space, then $H(X)$ is a completely metrizable topological group.*

However, whenever X is hemicompact rather than compact, Corollaries 1 and 2 are false, as we shall see in Examples 3–5. But first we look at some nice examples for contrast.

EXAMPLE 1. For each n , the space $E(\mathbf{R}^n, \mathbf{R}^n)$ is equal to $I(\mathbf{R}^n, \mathbf{R}^n)$, and is therefore completely metrizable.

PROOF. Let $f \in I(\mathbf{R}^n, \mathbf{R}^n)$. For each $m \in \mathbf{N}$, let A_m be the closed ball in $\mathbf{R}^n$ of radius m centered at the origin. Then for each m , $f|_{A_m}$ is an embedding. Because of invariance of domain, $f(\text{int } A_m)$ is open in $\mathbf{R}^n$ and hence open in $f(\mathbf{R}^n)$.

EXAMPLE 2. For each n , the space $H(\mathbf{R}^n)$ is a G_δ -set in $E(\mathbf{R}^n, \mathbf{R}^n)$, and is therefore completely metrizable.

PROOF. For each n , define

$$G_n = \{ f \in E(\mathbf{R}^n, \mathbf{R}^n) | A_n \subseteq f(\mathbf{R}^n) \},$$

where each A_n is the closed ball in $\mathbf{R}^n$ of radius n centered at the origin. It is clear that $H(\mathbf{R}^n) = \bigcap \{ G_n | n \in \mathbf{N} \}$. To see that each G_n is open, let $f \in G_n$. Then $f^{-1}(A_n)$ is compact, so that there exists an $m \in \mathbf{N}$ such that $f^{-1}(A_n) \subseteq \text{int } A_m$. Now $[[A_{m+1} - \text{int } A_m, \mathbf{R}^n - A_n]]$ is a basic neighborhood of f which, because of properties of $\mathbf{R}^n$, must be contained in G_n .

In order to obtain some examples which are not completely metrizable, it will suffice to find examples which are of first category in themselves. So in the following lemma we establish a condition which will help us obtain first category function spaces.

LEMMA. *Let X be a locally compact space, and let (Y, d) be a metric space. Let F be a subspace of $E(X, Y)$ having the following property: there exists an $x_0 \in X$ such that for every $\varepsilon > 0$, for every compact $A \subseteq X$, and for every $g \in F$, there exists an $f \in F$ such that $d(f(x), f(x_0)) < \varepsilon$ for some $x \in X - A$ and $d(f(a), g(a)) < \varepsilon$ for every $a \in A$. Then F is of first category in itself.*

PROOF. Let U be an open neighborhood of x_0 having compact closure. For each $n \in \mathbb{N}$, define

$$F_n = \{f \in F \mid d(f(x), f(x_0)) \geq 1/n \text{ for all } x \in X - U\}.$$

To see that $\bigcup \{F_n \mid n \in \mathbb{N}\} = F$, let $f \in F$. Now $f(U)$ is open in $f(X)$, so there exists an open V in Y such that $V \cap f(X) = f(U)$. Also there is an $n \in \mathbb{N}$ such that $B(f(x_0), 1/n) \subseteq V$. If $x \in X - U$, then $f(x) \notin f(U)$. But then $f(x) \notin B(f(x_0), 1/n)$, so that $f \in F_n$.

To see that each F_n is closed in F , let $n \in \mathbb{N}$ and let $f \in F - F_n$. Then there exists an $x \in X - U$ such that $d(f(x), f(x_0)) < 1/n$. Define $\delta = 1/n - d(f(x), f(x_0))$, and define

$$W = \left[[x_0, B(f(x_0), \delta/2)] \right] \cap \cdots \cap \left[[x, B(f(x), \delta/2)] \right],$$

which is a neighborhood of f . If $g \in W$, then

$$\begin{aligned} d(g(x), g(x_0)) &\leq d(g(x), f(x)) + d(f(x), f(x_0)) + d(f(x_0), g(x_0)) \\ &< \delta/2 + d(f(x), f(x_0)) + \delta/2 = 1/n. \end{aligned}$$

Therefore $g \in F - F_n$, so that F_n is closed.

Finally, to see that each $F - F_n$ is dense in F , let $n \in \mathbb{N}$ and let $W = [[A_1, V_1]] \cap \cdots \cap [[A_k, V_k]]$ be a basic open subset of F containing some element g . There exists some $\varepsilon < 1/n$ such that for each $i = 1, \dots, k$, $B(g(A_i), \varepsilon) \subseteq V_i$. Define $A = \bar{U} \cup A_1 \cup \cdots \cup A_k$. Then by hypothesis, there exists an $f \in F$ and an $x \in X - A$ such that $d(f(x), f(x_0)) < \varepsilon$ and $d(f(a), g(a)) < \varepsilon$ for all $a \in A$. Therefore $f \in W - F_n$, so that F_n is nowhere dense in F .

EXAMPLE 3. The space $E(\mathbb{R}, \mathbb{R}^2)$ is of first category in itself.

PROOF. Let x_0 be any element of $\mathbb{R}$, let $\varepsilon > 0$, let A be a compact subset of $\mathbb{R}$, and let $g \in E(\mathbb{R}, \mathbb{R}^2)$. Let $a < b < c$ be elements of $\mathbb{R}$ such that $[a, c] \subseteq \mathbb{R} - (A \cup \{x_0\})$, and let $y \in B(g(x_0), \varepsilon) - g(A)$. There exists an arc $\alpha: [a, b] \rightarrow \mathbb{R}^2 - g(A)$ such that $\alpha(a) = g(a)$ and $\alpha(b) = y$, and there exists an arc $\beta: [b, c] \rightarrow \mathbb{R}^2 - (g(A) \cup \alpha([a, b]))$ such that $\beta(b) = y$ and $\beta(c) = g(c)$. Then define $f \in E(\mathbb{R}, \mathbb{R}^2)$ by $f(x) = \alpha(x)$ if $x \in [a, b]$, $f(x) = \beta(x)$ if $x \in [b, c]$, and $f(x) = g(x)$ otherwise. Therefore the property in the Lemma is satisfied, so that $E(\mathbb{R}, \mathbb{R}^2)$ is of first category.

EXAMPLE 4. If $\{K_n \mid n \in \mathbb{N}\}$ is a pairwise disjoint family of copies of the Cantor set, then the space $K = \sum_n K_n$ is a hemicompact metrizable space such that $E(K, K)$ and $H(K)$ are both of first category in themselves.

PROOF. Let x_0 be any element of K , let $\varepsilon > 0$, let A be a compact subset of K , and let $g \in E(K, K^2)$ ($g \in H(K)$, respectively). We may assume that we can find an $x_1 \in K - A$ such that $g(x_1) \neq g(x_0)$. Let C_1 be an open subset of $B(g(x_0), \varepsilon/2) - \{g(x_1)\}$ which is a Cantor set, and let C_2 be an open subset of $K - (g(A) \cup C_1)$ containing $g(x_1)$ which is also a Cantor set. Let α be a homeomorphism from $C_1 \cup C_2$ onto $C_1 \cup C_2$ such that $(C_1 \cup \{x_1\}) \subseteq C_1$. Now let $h \in H(K)$ be defined by $h(x) = \alpha(x)$ if $x \in C_1 \cup C_2$ and $h(x) = x$ otherwise. Then define f by $f = h \circ g$, which is an element of $E(K, K)$ (an element of $H(K)$, respectively). The conclusion now follows from the Lemma.

We will give one more example showing that $E(X, X)$ and $H(X)$ may have different properties. It will be useful to have the following fact to use on this example.

PROPOSITION. *If $\{X_n | n \in \mathbb{N}\}$ is a family of pairwise disjoint copies of some connected space, then $H(\sum_n X_n)$ is homeomorphic to $\mathbf{P} \times \prod_n H(X_n)$, where $\mathbf{P}$ is the space of irrationals.*

PROOF. For each $m, n \in \mathbb{N}$, let $\alpha_m^n: X_m \rightarrow X_n$ be a homeomorphism. Define

$$\Phi: H\left(\sum_n X_n\right) \rightarrow H(\mathbb{N}) \times \prod_n H(X_n)$$

as follows. Let $h \in \sum_n X_n$. Then take

$$\Phi(h) = \langle \phi, \langle \alpha_{\phi(n)}^n \circ (h|_{X_n}) \rangle_n \rangle,$$

where $\phi \in H(\mathbb{N})$ is defined by: $\phi(n)$ is that element of $\mathbb{N}$ such that $h(X_n) = X_{\phi(n)}$. It is not difficult to check that Φ is a bijection.

Let $\pi: H(\mathbb{N}) \times \prod_n H(X_n) \rightarrow H(\mathbb{N})$ and, for each $m \in \mathbb{N}$, let $\pi_m: H(\mathbb{N}) \times \prod_n H(X_n) \rightarrow H(X_m)$ be the projection maps. Now $\pi \circ \phi$ is continuous since for each $n \in \mathbb{N}$, $\phi \in H(\mathbb{N})$, and $x \in X_n$, it is the case that $(\pi \circ \phi)([x, X_{\phi(n)}]) \subseteq [\{n\}, \{\phi(n)\}]$. Also each $\pi_m \circ \phi$ is continuous since for each compact $A \subseteq X_m$, each open $W \subseteq X_m$, and each $\phi \in H(\mathbb{N})$, we have $(\pi_m \circ \phi)([A, (\alpha_{\phi(m)}^m)^{-1}(W)]) \subseteq [A, W]$. Therefore Φ is continuous.

To see that Φ^{-1} is continuous let $[A, W]$ be a subbasic open set in $H(\sum_n X_n)$ and let $\phi \in H(\mathbb{N})$. There is some $N \in \mathbb{N}$ such that $A \subseteq X_1 \cup \dots \cup X_N$. For each $n = 1, \dots, N$, let $A_n = A \cap X_n$ and $W_n = W \cap X_{\phi(n)}$. Then

$$\begin{aligned} \pi^{-1}([\{1\}, \{\phi(1)\}]) \cap \dots \cap [\{N\}, \{\phi(N)\}]) \\ \cap \pi_1^{-1}([A_1, W_1]) \cap \dots \cap \pi_N^{-1}([A_N, W_N]) \subseteq ([A, W]), \end{aligned}$$

so that Φ^{-1} is continuous.

Now we show that $H(\mathbb{N})$ is a G_δ -set in $E(\mathbb{N}, \mathbb{N}) = I(\mathbb{N}, \mathbb{N})$, which is completely metrizable. First note that for each n , $F_n \equiv [\mathbb{N}, \mathbb{N} - \{n\}]$ is a closed subset of $E(\mathbb{N}, \mathbb{N})$ contained in $E(\mathbb{N}, \mathbb{N}) - H(\mathbb{N})$. If $f \in E(\mathbb{N}, \mathbb{N}) - H(\mathbb{N})$, then there exists an n such that $f \in F_n$. Therefore $E(\mathbb{N}, \mathbb{N}) - H(\mathbb{N}) = \bigcup \{F_n | n \in \mathbb{N}\}$, so that $H(\mathbb{N})$ is a G_δ -set in $E(\mathbb{N}, \mathbb{N})$. Finally, $H(\mathbb{N})$ is homeomorphic to $\mathbf{P}$ since every separable zero-dimensional completely metrizable space with no nonempty compact open subset is homeomorphic to $\mathbf{P}$ (see [1]).

EXAMPLE 5. If

$$Z = \{0\} \cup \left(\bigcup \{(n, n+1) \cup [1/2^{2n}, 1/2^{2n-1}] \subseteq \mathbf{R} | n \in \mathbb{N}\} \right)$$

with the usual topology, then Z is a hemicompact metric space such that $E(Z, Z)$ is of first category in itself while $H(Z)$ is completely metrizable.

PROOF. For each n , let $R_n = (n, n+1)$, let $R = \bigcup \{R_n | n \in \mathbb{N}\}$, and let $C = Z - R$. From the Proposition it follows that $H(R)$ is homeomorphic to $\mathbf{P} \times \prod_n H(R_n)$, so that $H(Z)$ is homeomorphic to $H(C) \times \mathbf{P} \times \prod_n H(R_n)$, which is completely metrizable. On the other hand, it follows from the Lemma that $E(Z, Z)$

is of first category. To see this, let $\varepsilon > 0$, let A be a compact subset of Z , and let $g \in E(Z, Z)$. For each n , define

$$A_n = \{0\} \cup \left(\bigcup \{ [1/2^{2i}, 1/2^{2i-1}] \mid i \geq n \} \right).$$

Now there is some n such that $1/2^{2n-1} < \varepsilon$, $R_n \cap g(A) = \emptyset$, and $R_n \cap g(Z) \neq \emptyset$. Let α be a homeomorphism from A_n onto A_{n+1} , and let β be an embedding of R_n into $[1/2^{2n}, 1/2^{2n-1}]$. Let $h \in E(Z, Z)$ be defined by $h(z) = \alpha(z)$ if $z \in A_n$, $h(z) = \beta(z)$ if $z \in R_n$, and $h(z) = z$ otherwise. Finally take $f = h \circ g$.

These examples suggest the following questions for a hemicompact metric space X . If $E(X, X)$ is completely metrizable, is $H(Z)$ completely metrizable? If $H(X)$ is of second category in itself, must it be completely metrizable? Is there a convenient characterization for $H(X)$ being completely metrizable?

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