ABSOLUTE TOTAL-EFFECTIVENESS OF A TOTAL EFFECTIVE (N, p_n) METHOD

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ABSTRACT. In the present paper we investigate absolute total effectiveness of a (N, p_n) method. It is established that a method known to be total effective is also absolute total effective.

1. Let $\{p_n\}$ be a sequence of constants, real or complex, such that $P_n = p_0 + p_1 + \dots + p_n \neq 0$ for $n \geq 0$. The sequence $t_n(A(x))$, where

$$t_n(A(x)) = (1/P_n) \sum_{k=0}^n P_{n-k} A_k(x),$$

is called the (N, p_n) mean of the series $\sum_{n=0}^{\infty} A_n(x)$. An (N, p_n) method is said to be $(A(x), E_f)$ -effective or $|A(x), E_f|$ -effective according as $\lim_{n \rightarrow \infty} t_n(A(x))$ exists and is finite or $\sum_{n=1}^{\infty} |t_n(A(x)) - t_{n-1}(A(x))| < \infty$ for each $x \in E_f$.

We write

$$\phi_1(t) = t^{-1} \int_0^t \frac{1}{2} \{f(x+t) + f(x-t)\} dt;$$

$$\psi(t) = \frac{1}{2} \{f(x+t) - f(x-t)\}; \quad h(t) = \psi(t)/t;$$

$$P(n, k) = P_{n-k}/P_n - P_{n-k-1}/P_{n-1} \quad \text{for } 0 \leq k \leq n > 0; P(0, 0) = 1;$$

K denotes a positive constant, not necessarily the same at each occurrence; $\{R_n\} \equiv \{(n+1)p_n/P_n\} \in B$ means $|R_n| \leq K$ for all n ; $\Delta_n a_n = \Delta a_n = a_n - a_{n+1}$; $p_n = 0$ if $n < 0$.

We denote Fourier series of an L -integrable periodic function $f(t)$ with period 2π by $F(t)$; conjugate series of $F(t)$ by $\tilde{F}(t) \equiv \sum_{n=1}^{\infty} B_n(t)$; term-by-term differential of $F(t)$ by $F'(t)$; the sequence $\{nB_n(t)\}$ by $F^*(t)$.

A point x is said to be

a(i) (F_1) -regular, (F_1^*) -regular or (F'_1) -regular if $\phi_1(t)$, $\psi(t)$ or $h(t)$ tends to some finite limit as $t \rightarrow 0$ respectively;

(ii) $(\tilde{F}_1)$ -regular if $\lim_{\epsilon \rightarrow +0} \int_{\epsilon}^{\pi} \{\psi(t)/\tan \frac{1}{2}t\} dt$ exists;

b(i) $|F_1|$ -regular, $|F_1^*|$ -regular or $|F'_1|$ -regular if $\phi_1(t)$, $\psi(t)$ or $h(t)$ is of bounded variation in $(0, \pi)$ respectively;

(ii) $|\tilde{F}_1|$ -regular if $\int_0^{\pi} |h(t)| dt < \infty$.

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Let Y denote any one of the symbols: F , $\tilde{F}$, F^* and F' , and $E(Y_1, f)$ and $E|Y_1, f|$ denote (Y_1) - and $|Y_1|$ -regular points. Then a $(A(x), E_f)$ -effective method (N, p_n) is said to be (Y_1) -effective for each fixed Y , if $A(x) = Y(x)$ and $E_f = E(Y_1, f)$. The method (N, p_n) which is $|A(x), E_f|$ -effective is said to be $|Y_1|$ -effective if $A(x) = Y(x)$ and $E_f = E|Y_1, f|$. The (N, p_n) method is said to be total-effective if it is (F_1) -, $(\tilde{F}_1)$ -, (F_1^*) - and (F_1') -effective simultaneously and is said to be absolute total-effective if it is $|F_1|$ -, $|\tilde{F}_1|$ -, $|F_1^*|$ - and $|F_1'|$ -effective simultaneously.

Concerning the total-effectiveness of the (N, p_n) method the following ([1], see also [3, Theorem A]) is known.

THEOREM A. *Let $\{p_n\}$ be such that, for $n = 1, 2, \dots$,*

$$(1.1) \quad n \sum_{k=1}^n k^{-2} |P_k| \leq K |P_n|$$

and

$$(1.2) \quad n \sum_{k=1}^n k |\Delta^2 p_{k-2}| \leq K |P_n|.$$

Then the (N, p_n) method is total-effective.

For the $|F_1|$ -effectiveness of the (N, p_n) method we have [3, Theorem D] the following.

THEOREM B. *If $\{p_n\}$ is such that $\{R_n\} \in BV$*

$$(1.3) \quad \sum_{k=n}^{\infty} 1/|P_k| \leq Kn/|P_n|, \quad n = 1, 2, \dots,$$

and

$$(1.4) \quad p_0 > 0, \quad \{p_{n+1} - p_n\} \text{ is positive nonincreasing,}$$

then the (N, p_n) method is $|F_1|$ -effective.

In this direction Izumi and Izumi [6, Theorem 2] proved the following.

THEOREM C. *Suppose $\{p_n\}$ satisfies condition (1.4) and that*

$$(1.5) \quad \sum_{k=1}^n 1/P_k < \infty.$$

Then the (N, p_n) method is $|F_1|$ -effective if and only if

$$(1.6) \quad \sum_{n=k}^{\infty} |P(n, k)| \leq K \quad \text{for all } k \geq 1.$$

We see that for the sequence satisfying (1.4) (cf. [4, p. 104])

$$\sum_{k=1}^n k |\Delta^2 p_{k-2}| = |\Delta^2 p_{-1}| - \sum_{k=2}^n k \Delta^2 p_{k-2} \leq K + p_n.$$

Hence (1.2) is true, since $\{R_n\} \in B$ (see [6, (8)]). In view of (2.2), $\{R_n\} \in B$ and Lemma 4 we find that (1.6) is true whenever (1.4) holds. This shows that the

necessity part of Theorem C has no force. As for the sufficiency we see that in the course of the proof of the theorem, as given in [6, p. 119], the authors assume $p_1 = p_0$ and (1.4) holds for $n > 0$. But in the proof [6, p. 121] the condition is used for $n \geq 0$; consequently $0 \leq p_{n+1} - p_n \leq p_1 - p_0 = 0$, that is $p_n = p_0$ for all n . This gives that the hypothesis (1.5) is not meaningful. However, a particular case of the theorem proved in this paper shows the result holds good when $p_1 = p_0$ if we assume (1.3), or equivalently (1.1) by Lemma 2, in place of (1.5). Condition (1.3) appears to be indispensable as it is a necessary condition for F_1 -effectiveness of the (N, p_n) method whenever (1.2) holds [5, Theorem 2].

Absolute total-effectiveness, inter alia $|F_1|$ -effectiveness, of the (N, p_n) method for the sequences which are not necessarily monotonic is given [4, Theorem A'] in the following theorem.

THEOREM D. *Let $\{p_n\}$ satisfy the conditions (1.2), (1.3) and that $\{R_n\} \in BV$. Then the (N, p_n) method is absolute total-effective.*

In view of the foregoing results we investigate absolute total-effectiveness of the (N, p_n) method for a more general sequence. Consequently we prove that the summability method of Theorem A is also absolute total-effective. We prove the following.

THEOREM. *The (N, p_n) method is absolute total-effective if $\{p_n\}$ satisfies the conditions (1.1) and (1.2).*

2. For the proof of our theorem we need the following results.

LEMMA 1. *Let $\{a_n\}$ be a given sequence; then for $n \geq r \geq 0$*

$$(1-x) \sum_{k=r}^n a_k x^k = a_r x^r - a_{n+1} x^{n+1} - \sum_{k=r}^n (\Delta a_k) x^{k+1}.$$

The proof of Lemma 1 is direct. The following lemma is contained in [4, Lemma 3].

LEMMA 2. *Let the sequence $\{p_n\}$ be such that $|P_{2n}| \sim |P_n|$ or $\{R_n\} \in B$. Then (1.1), (1.3) and that there is a $\delta > 0$ such that uniformly in $1 \leq k \leq n$*

$$(2.1) \quad |P_k|/|P_n| = O((k/n)^{1+\delta})$$

are all equivalent.

LEMMA 3. *Let $\{p_k\}$ satisfy conditions (1.1) and (1.2). Then*

$$(i) \quad n \sum_{k=1}^n |\Delta p_{k-1}| \leq K|P_n| \Rightarrow \sum_{k=1}^n k|\Delta p_{k-1}| \leq K|P_n|$$

$$\Rightarrow \{R_n\} \in B \quad \text{and} \quad P_n^* = \sum_{k=0}^n |p_k| \leq K|P_n|,$$

$$(ii) \quad n^2 |\Delta^2 p_{n-1}| \leq K|P_n|.$$

PROOF. The first part of (i) is given in [3, (2.4')]; whereas the second is direct. The last part follows from the equalities

$$(2.2) \quad (n+1)p_n = P_n - \sum_{k=0}^n k \Delta p_{k-1}$$

and

$$\begin{aligned} P_n^* &= \sum_{k=1}^n k(|p_{k-1}| - |p_k|) + (n+1)|p_n| \\ &\leq \sum_{k=1}^n k|\Delta p_{k-1}| + (n+1)|p_n|. \end{aligned}$$

That (ii) holds follows from the fact that $\{R_n\} \in B$ and that

$$\sum_{k=0}^n k^2 \Delta^2 p_{k-2} = 2P_n - (n+1)^2 \Delta p_{n-1} - (2n+3)p_n.$$

LEMMA 4. If $\sum_{k=1}^n k|\Delta p_{k-1}| \leq K|P_n|$, then the (N, p_n) method is absolutely regular viz. $p_n = o(P_n)$ and (1.6) holds.

Lemma 4 has been proved in [2, Lemma 4]. A different proof can be obtained from [7, p. 95].

3. Proof of the theorem. $|F_1|$ -effectiveness. In order to prove this part it is sufficient to show (cf. [6, p. 119]) that, uniformly in $0 < t \leq \pi$,

$$(3.1) \quad S = \sum_{n=1}^{\infty} \left| \sum_{k=1}^n P(n, k) c_k(t) \right| < \infty,$$

where $c_k(t) = \int_0^t k u \sin ku \, du$.

We write $\tau = [2\pi/t]$. We have

$$\begin{aligned} S &\leq \sum_{n=1}^{\tau-1} \left| \sum_{k=1}^n P(n, k) c_k(t) \right| + \sum_{n=\tau}^{\infty} \left| \sum_{k=1}^{\tau-1} P(n, k) c_k(t) \right| \\ &\quad + \sum_{n=\tau}^{\infty} \left| \sum_{k=\tau}^n P(n, k) c_k(t) \right| \\ &= S_1 + S_2 + S_3, \quad \text{say.} \end{aligned}$$

Since $|\sin kt| \leq kt$, we get

$$|c_k(t)| = |-t \cos kt + k^{-1} \sin kt| \leq Kt.$$

Consequently, by Lemma 4

$$\begin{aligned} (3.2) \quad S_1 &\leq Kt \sum_{n=1}^{\tau} \sum_{k=1}^n |P(n, k)| = Kt \sum_{k=1}^{\tau} \sum_{n=k}^{\tau} |P(n, k)| \\ &\leq Kt \sum_{k=1}^{\tau} 1 \leq K. \end{aligned}$$

Similarly,

$$(3.3) \quad S_2 \leq Kt \sum_{k=1}^{\tau-1} \sum_{n=\tau}^{\infty} |P(n, k)| \leq K.$$

In order to prove $S_3 \leq K$, we write

$$(t + k^{-1} \exp i(\pi/2)) \exp(ikt) = u_k(t) \exp ikt,$$

and observe that $-c_k(t)$ is the real part of $u_k(t) \exp ikt$. First applying Lemma 1 and then using the fact that

$$\Delta_k P(n, k) = \Delta_k p_{n-k}/P_{n-1} - p_{n-k}p_n/(P_n P_{n-1})$$

and that, uniformly in $0 < t \leq \pi$,

$$|u_\tau(t)| \leq Kt, \quad |\Delta_k u_k(t)| \leq Kk^{-2}$$

we see that

$$\begin{aligned} J_n &= \left| \sum_{k=\tau}^n P(n, k) u_k(t) \exp ikt \right| \\ &\leq K\tau |P(n, \tau) u_\tau(t)| + K\tau \left| \sum_{k=\tau}^n P(n, k) \{\Delta_k u_k(t)\} \exp ikt \right| \\ &\quad + K\tau \left| \sum_{k=\tau}^n \{\Delta_k P(n, k)\} u_{k+1}(t) \exp ikt \right| \\ &\leq K|P(n, \tau)| + K\tau \sum_{k=\tau}^n |P(n, k)| k^{-2} \\ &\quad + \frac{K\tau}{|P_{n-1}|} \left| \sum_{k=\tau}^n (\Delta_k p_{n-k}) u_{k+1}(t) \exp ikt \right| \\ &\quad + \frac{K|p_n|\tau}{|P_n P_{n-1}|} \left| \sum_{k=\tau}^n p_{n-k} u_{k+1}(t) \exp ikt \right| \\ &= J_n(1) + J_n(2) + J_n(3) + J_n(4), \quad \text{say.} \end{aligned}$$

We now proceed to prove that, uniformly in $0 < t \leq \pi$,

$$\sum_{n=\tau}^{\infty} J_n(r) \leq K, \quad r = 1, 2, 3, 4.$$

Evidently, by Lemma 4

$$(3.4) \quad \sum_{n=\tau}^{\infty} J_n(1) = K \sum_{n=\tau}^{\infty} |P(n, \tau)| \leq K.$$

Again applying Lemma 4 after a change of order of summation, we get

$$(3.5) \quad \sum_{n=\tau}^{\infty} J_n(2) \leq K\tau \sum_{k=\tau}^{\infty} k^{-2} \leq K.$$

Separating real and imaginary parts of $u_k(t)$, we see that they both are nonnegative nonincreasing; hence Abel's lemma and the fact that $|u_\tau(t)| \leq Kt$ give

$$J_n(3) \leq \frac{K}{|P_{n-1}|} \max_{\tau \leq r \leq n} \left| \sum_{k=\tau}^r (\Delta_k p_{n-k}) \exp ikt \right| = \frac{KJ_n^*(3)}{|P_{n-1}|},$$

say. But

$$\begin{aligned} J_n^*(3) &= \max_{\tau \leq r \leq n} \left| \sum_{k=n-r}^{n-\tau} (\Delta p_k) \exp i(n-k)t \right| \\ &\leq \sum_{k=0}^{\tau} |\Delta p_k| + \max_{\tau \leq r \leq n-\tau} \left| \sum_{k=r}^{n-\tau} \Delta p_k \exp i(n-k)t \right| \\ &\leq KP_\tau/\tau + K\tau \left\{ \sum_{k=\tau}^n |\Delta^2 p_k| + \max_{\tau \leq r \leq n} |\Delta p_r| \right\} \end{aligned}$$

by virtue of Lemmas 1 and 3. Again using Lemma 3, we get

$$\begin{aligned} \sum_{n=\tau}^{\infty} J_n(3) &\leq \frac{K|P_\tau|}{\tau} \sum_{n=\tau}^{\infty} \frac{1}{|P_n|} + K\tau \sum_{n=\tau}^{\infty} \frac{1}{|P_n|} \sum_{k=\tau}^n |\Delta^2 p_k| \\ &\quad + K\tau \sum_{n=\tau}^{\infty} \max_{\tau \leq r \leq n} (r^{-2}|P_r|/|P_n|). \end{aligned}$$

Applying Abel's transformation and (1.2) we see that

$$\begin{aligned} \sum_{k=\tau}^n |\Delta^2 p_k| &\leq \sum_{k=\tau}^n k^{-2} \sum_{j=\tau}^k j |\Delta^2 p_j| + n^{-1} \sum_{j=\tau}^n j |\Delta^2 p_j| \\ &\leq \sum_{k=\tau}^n k^{-3} |P_k| + n^2 |P_n|. \end{aligned}$$

This and Lemma 2 give that

$$\begin{aligned} \sum_{n=\tau}^{\infty} J_n(3) &\leq K + K\tau \sum_{n=\tau}^{\infty} \frac{1}{|P_n|} \sum_{k=\tau}^n k^{-3} |P_k| + K\tau \sum_{n=\tau}^{\infty} n^{-2} \\ &\quad + K\tau \sum_{n=\tau}^{\infty} n^{-1-\delta} \max_{\tau \leq r \leq n} r^{\delta-1}. \end{aligned}$$

Choosing a positive number α such that $\alpha < \min(1, \delta)$, we see that max of $r^{\delta-1}$ for $\tau \leq r \leq n$ does not exceed

$$n^{\delta-\alpha} (r/n)^{\delta-\alpha} r^{\alpha-1} \leq n^{\delta-\alpha} \tau^{\alpha-1}.$$

Hence by a change of order of summation and Lemma 2 we obtain

$$(3.6) \quad \sum_{n=\tau}^{\infty} J_n(3) \leq K + \tau \sum_{k=\tau}^{\infty} k^{-3} |P_k| \sum_{n=k}^{\infty} 1/|P_n| + \tau^\alpha \sum_{n=\tau}^{\infty} n^{-1-\alpha} \leq K.$$

Now we estimate the sum involving $J_n(4)$. Abel's transformation gives

$$\begin{aligned} \max_{\tau \leq r \leq n} \left| \sum_{k=\tau}^r p_{n-k} \exp ikt \right| &\leq \sum_{k=n-\tau+1}^n |p_{n-k}| + \max_{\tau \leq r \leq n-\tau} \left| \sum_{k=\tau}^r p_{n-k} \exp ikt \right| \\ &\leq KP_{\tau}^* + K\tau \left\{ \sum_{k=\tau}^n |\Delta_k p_{n-k}| + \max_{\tau \leq r \leq n-\tau} |p_r| \right\} \\ &\leq KP_{\tau}^* + K\tau \left\{ n^{-1} |P_n| + \max_{\tau \leq r \leq n} r^{-1} |P_r| \right\}, \end{aligned}$$

by Lemma 3. Using this result after an application of Abel's lemma we see that

$$(3.7) \quad \sum_{n=\tau}^{\infty} J_n(4) \leq K\tau \sum_{n=\tau}^{\infty} n^{-2} + K|P_{\tau}| \sum_{n=\tau}^{\infty} n^{-1}/|P_n| + K \leq K,$$

by virtue of Lemma 2.

Combining (3.4)–(3.7) we establish $S_3 \leq K$. This together with (3.2) and (3.3) gives (3.1). This completes the proof for $|F_1|$ -effective.

In order to prove the remaining parts of the theorem we first observe that uniformly in $0 < t \leq \pi$

$$(3.8) \quad t \sum_{n=1}^{\infty} \left| \sum_{k=1}^n P(n, k) \exp ikt \right| < \infty.$$

For if $c_k(t)$ is replaced by $t \exp ikt$ in (3.1), then following the proof for (3.1), it is easy to see that (3.8) is true. That (3.8) implies $|\tilde{F}_1|$ -, $|F_1^*|$ - and $|F_1'|$ -effectiveness is shown in [3, pp. 199–220].

This completes the proof of the theorem.

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