WHEN TOTAL VARIATION IS ADDITIVE

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ABSTRACT. Let f and g be continuous functions of bounded variation on $[0, 1]$. We use the Dini derivatives of f and g to give a necessary and sufficient condition that the equation $V(f + g) = V(f) + V(g)$ holds.

Let f and g be continuous functions of bounded variation on $[0, 1]$. We know that $V(f + g) \leq V(f) + V(g)$ where V denotes total variation on $[0, 1]$. Equality holds in some special cases, for example when f and g are both nondecreasing or both nonincreasing. On the other hand, equality does not hold when f is nonconstant and $g = -f$. In this note we give a necessary and sufficient condition for equality to hold in terms of the four Dini derivatives, D^+ , D_+ , D^- , D_- , of f and g .

DEFINITION 1. We say that f is increasing (or decreasing) at a point $x \in (0, 1)$ if

$$\min(D^+ f(x), D_+ f(x), D^- f(x), D_- f(x)) > 0$$

(or $\max(D^+ f(x), D_+ f(x), D^- f(x), D_- f(x)) < 0$).

We offer the following

THEOREM 1. Let E_f denote the set of points where f is increasing and g is decreasing and E_g denote the set of points where g is increasing and f is decreasing. Then a necessary and sufficient condition for

$$(*) \quad V(f + g) = V(f) + V(g)$$

to hold is that there exist sets S_f and S_g such that $E_f \cup E_g = S_f \cup S_g$ and $\lambda f S_f = \lambda g S_g = 0$, where λ is Lebesgue outer measure. Moreover, a necessary condition for $(*)$ to hold is that $\lambda(E_f \cup E_g) = 0$.

The idea is that $(*)$ holds when there are not "too many" points where one function increases and the other decreases. Before proving Theorem 1, let us discuss some of its consequences. If f is absolutely continuous and g is singular, then $\lambda(E_f \cup E_g) = 0$ because $g' = 0$ almost everywhere. Then $\lambda f(E_f \cup E_g) = 0$ and $(*)$ holds. Thus $(*)$ might hold even when f is strictly increasing and g is strictly decreasing on $[0, 1]$. Just make one function absolutely continuous and the other singular. Of course $(*)$ cannot hold if f is increasing and g is decreasing at each point of some subinterval of $[0, 1]$.

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The necessary condition $\lambda(E_f \cup E_g) = 0$ is not in general sufficient. Consider a singular function f and $g = -f$.

Another familiar example of equation (*) occurs when f and g are the Jordan decomposition of a function of bounded variation, $f + g$, into the sum of two monotone functions.

If $f = f_1 + f_2$, $g = g_1 + g_2$ where f_1, g_1 are absolutely continuous and f_2, g_2 are singular, then (*) holds for f and g if and only if (*) holds for f_1 and g_1 and for f_2 and g_2 .

When f and g are absolutely continuous, (*) holds if and only if $f'g' \geq 0$ almost everywhere. This also follows from the equation $V(f) = \int_0^1 |f'|$, etc.

We begin with two lemmas. The first is our only result that does not require bounded variation.

LEMMA 1. *Let v be a real valued function on $(0, 1)$ which is increasing (or decreasing) at each point of a set $S \subset (0, 1)$, and let $\lambda v S = 0$. Then $\lambda S = 0$.*

PROOF. We prove the case where v is increasing at each point of S . We have $S = \bigcup_{ink} S_{ink}$ where

$$S_{ink} = \{x: (i-1)k^{-1} \leq x < ik^{-1}, |v(x)| < n, \\ (v(x+h) - v(x))h^{-1} > n^{-1} \text{ for } |h| < k^{-1}\}.$$

It suffices to prove that each $\lambda S_{ink} = 0$. But v is nondecreasing on S_{ink} . Put

$$w(x) = \inf(\inf v([x, 1) \cap S_{ink}), \sup v S_{ink}).$$

Then w is nondecreasing on $(0, 1)$. Also $w' \geq n^{-1}$ almost everywhere on S_{ink} , w is continuous at all but at most countably many points of S_{ink} , $w S_{ink} = v S_{ink}$, and $\lambda w S_{ink} = 0$. It follows from [1, pp. 126, 100] that $\lambda S_{ink} = 0$. $\square$

In what follows, $F(x) = V_0^x(f)$ and $G(x) = V_0^x(g)$ for $0 \leq x \leq 1$. Thus F and G are continuous and nondecreasing. Let f^* and F^* denote the signed Borel measure and the Borel measure induced by the functions f and F respectively. Define g^* and G^* analogously. For any interval J , the diameter of the set $f(F^{-1}J)$ cannot exceed the length of J . Routine arguments show that $\lambda f A \leq \lambda F A$ for any set $A \subset (0, 1)$. In particular $\lambda f A = 0$ if $\lambda F A = 0$. We need the converse of this last statement.

LEMMA 2. *Let $A \subset (0, 1)$ such that $\lambda f A = 0$. Then $\lambda F A = F^* A = 0$.*

PROOF. For any subinterval (a, b) of $(0, 1)$ we define the function $s((a, b), y) =$ cardinality of the set $(a, b) \cap f^{-1}y$ for all real y . Then from [1, p. 280] we see that s is nonnegative, measurable, summable and

$$\int s((a, b), y) dy = V_a^b(f) = F(b) - F(a) = \lambda F(a, b).$$

In particular, $\int s((0, 1), y) dy = V(f) < \infty$. Take any $c > 0$. There is an open set U such that $f A \subset U$ and $\int_U s((0, 1), y) dy < c$. Let I_n , $n = 1, 2, 3, \dots$, be the component

intervals of the open set $(0, 1) \cap f^{-1}U$. Then $\sum_n s(I_n, y) \leq s((0, 1), y)$ and

$$\begin{aligned} \lambda F\left(\bigcup_n I_n\right) &= \sum_n \lambda F I_n = \sum_n \int s(I_n, y) dy = \int \sum_n s(I_n, y) dy \\ &\leq \int_U s((0, 1), y) dy < c. \end{aligned}$$

But $A \subset \bigcup_n I_n$, so $\lambda FA < c$. Since c is arbitrary, $\lambda FA = 0$. Finally, $F^*A = \lambda FA = 0$ by [1, p. 100]. $\square$

We are now ready for the proof of Theorem 1.

PROOF OF SUFFICIENCY. Assume that $E_f \cup E_g$ can be covered by two sets S as described. Without loss of generality we assume that the sets S_f and S_g are Borel sets; note that if $A \supset fS_f$ is a G_δ -set with $\lambda A = 0$, then $f^{-1}A$ is a Borel set and $\lambda f(f^{-1}A) = 0$. Certainly E_f and E_g are Borel sets because f and g are continuous.

Put

$$U_f = \{x: f \text{ is finitely or infinitely differentiable at } x \text{ and } f'(x) > 0\},$$

$$W_f = \{x: f \text{ is finitely or infinitely differentiable at } x \text{ and } f'(x) < 0\}.$$

Again U_f and W_f are Borel sets. We define U_g and W_g analogously for g . If $f' = 0$ everywhere on the Borel set B , then $f^*B = 0$ [1, p. 127]. Thus $F^*B = 0$ also. It follows from this and from [1, pp. 126, 127] that $f^* \geq 0$ on any subset of U_f , $f^* \leq 0$ on any subset of W_f , and $F^* = f^* = 0$ on any subset of $[0, 1] \setminus (U_f \cup W_f)$. Likewise $g^* \geq 0$ on any subset of U_g , $g^* \leq 0$ on any subset of W_g , and $G^* = g^* = 0$ on any subset of $[0, 1] \setminus (U_g \cup W_g)$.

We note that $(U_f \setminus S_f) \cap (W_g \setminus S_g) = (U_f \cap W_g) \setminus (S_f \cup S_g) = \emptyset$ because $U_f \cap W_g \subset E_f$. Likewise $(U_g \setminus S_g) \cap (W_f \setminus S_f) = \emptyset$ because $U_g \cap W_f \subset E_g$. Moreover $F^*S_f = G^*S_g = f^*S_f = g^*S_g = 0$ by Lemma 2. From this it follows that

$$\begin{aligned} V(f + g) &\geq f^*(U_f \setminus S_f) - f^*(W_f \setminus S_f) + g^*(U_g \setminus S_g) - g^*(W_g \setminus S_g) \\ &= f^*U_f - f^*W_f + g^*U_g - g^*W_g \\ &= F^*(U_f \cup W_f) + G^*(U_g \cup W_g) \\ &= V(f) + V(g). \quad \square \end{aligned}$$

PROOF OF NECESSITY. Assume that $V(f + g) = V(f) + V(g)$. Then $V(f + g) = f^*U_f - f^*W_f + g^*U_g - g^*W_g$, where we use the same notation employed in the proof of sufficiency. Partition $[0, 1]$ into the union of two disjoint Borel sets A and B such that the set function $(f + g)^* = f^* + g^*$ is nonnegative on all subsets of A and nonpositive on all subsets of B . Fix a Borel set $C \subset U_f \cap W_g \cap B$. Let t be the nondecreasing function on $[0, 1]$ for which $t^*D = f^*D$ for $D \subset C$ and $t^*D = 0$ for $D \cap C = \emptyset$. Then $(f^* - t^*)D = f^*D - f^*D = 0$ for $D \subset C$, and $f^* - t^* \geq 0$ on subsets of U_f and $f^* - t^* \leq 0$ on subsets of W_f . Also $(g^* + t^*)D = g^*D + f^*D \leq 0$ for $D \subset C$ and $g^* + t^* \leq 0$ on subsets of W_g and $g^* + t^* \geq 0$ on subsets of U_g .

So

$$\begin{aligned}
 V(f+g) &\leq V(f-t) + V(g+t) \\
 &\leq f^*U_f - t^*C - f^*W_f + g^*U_g - t^*C - g^*W_g \\
 &= f^*U_f - f^*W_f + g^*U_g - g^*W_g - 2t^*C \\
 &= V(f) + V(g) - 2f^*C = V(f+g) - 2f^*C
 \end{aligned}$$

and $f^*C = 0$. It follows that $F^*(U_f \cap W_g \cap B) = 0$. We can prove in an analogous manner that $G^*(U_f \cap W_g \cap A) = 0$. Let S_1 be the set formed by adjoining to $U_f \cap W_g \cap B$ the set of all points where f is not finitely or infinitely differentiable. Likewise S_2 is formed by adjoining to $U_f \cap W_g \cap A$ the set of all points where g is not finitely or infinitely differentiable. Then $F^*S_1 = G^*S_2 = 0$ [1, p. 125], and $E_f \subset S_1 \cup S_2$. Likewise there are Borel sets S_3 and S_4 such that $F^*S_3 = G^*S_4 = 0$ and $E_g \subset S_3 \cup S_4$.

Put $T_f = S_1 \cup S_3$ and $T_g = S_2 \cup S_4$. Then

$$F^*T_f = G^*T_g = 0 = \lambda FT_f = \lambda GT_g \quad [1, p. 100].$$

Moreover $\lambda fT_f = \lambda gT_g = 0$, and $E_f \cup E_g \subset T_f \cup T_g$. Finally, put $S_f = (E_f \cup E_g) \cap T_f$ and $S_g = (E_f \cup E_g) \cap T_g$. The rest is clear. $\square$

To complete the proof of Theorem 1, assume that (*) holds. Then

$$\lambda(S_f \cap E_f) = \lambda(S_f \cap E_g) = \lambda(S_g \cap E_f) = \lambda(S_g \cap E_g) = 0 = \lambda E_f = \lambda E_g$$

by Lemma 1, and hence $\lambda(E_f \cup E_g) = 0$. $\square$

If f is nondecreasing and g is nonincreasing we can now state when f and g are the Jordan decomposition of the function of bounded variation $f+g$.

COROLLARY 1. *Let f be a continuous nondecreasing function and g be a continuous nonincreasing function on $[0, 1]$. Then f and g are the Jordan decomposition of the function $f+g$ if and only if there exist sets S_f and S_g such that $E_f = S_f \cup S_g$ and $\lambda fS_f = \lambda gS_g = 0$.*

PROOF. Theorem 1. $\square$

We find when F and $-G$ are the Jordan decomposition of $F-G$ in terms of the functions f and g .

COROLLARY 2. *Let F, G, f, g be as described before. Then*

$$(**) \quad V(f) + V(g) = V(F-G)$$

holds if and only if

$$(***) \quad V(f) + V(g) = V(f+g) = V(f-g)$$

holds.

PROOF. Let E be the set of all points where f is increasing or decreasing and g is increasing or decreasing. Let A be the set of all points where F is increasing and G is increasing. Put

$$B_1 = \{x: f \text{ and } F \text{ are finitely or infinitely differentiable and } |f'(x)| = F'(x) \neq 0\},$$

$$B_2 = \{x: g \text{ and } G \text{ are finitely or infinitely differentiable and } |g'(x)| = G'(x) \neq 0\}.$$

By [1, pp. 127, 271],

$$\lambda f([0, 1] \setminus B_1) = \lambda g([0, 1] \setminus B_2) = \lambda F([0, 1] \setminus B_1) = \lambda G([0, 1] \setminus B_2) = 0.$$

Suppose (***) holds. By Theorem 1 there exist sets S_1 and S_2 such that $B_1 \cap B_2 \subset S_1 \cup S_2$ and $\lambda f S_1 = \lambda g S_2 = 0$. By Lemma 2, $\lambda F S_1 = \lambda G S_2 = 0$. Then

$$\lambda F(S_1 \cup (A \setminus B_1)) = \lambda G(S_2 \cup (A \setminus B_2)) = 0,$$

and by Theorem 1, (**) holds.

Now suppose (**) holds. By Theorem 1 there exist sets S_3 and S_4 such that $B_1 \cap B_2 \subset S_3 \cup S_4$ and $\lambda F S_3 = \lambda G S_4 = 0$. Then $\lambda f S_3 = \lambda g S_4 = 0$ and

$$\lambda f(S_3 \cup (E \setminus B_1)) = \lambda g(S_4 \cup (E \setminus B_2)) = 0.$$

By Theorem 1, (***) holds. $\square$

Corollary 2 can also be proved by other means, but we will not do that here.

REFERENCES

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