INTERPOLATION OF UNIFORMLY CONVEX BANACH SPACES

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ABSTRACT. If A_0 and A_1 are a compatible couple of Banach spaces, one of which is uniformly convex, then the complex interpolation spaces $[A_0, A_1]_\theta$ are also uniformly convex for $0 < \theta < 1$. Estimates are given for the moduli of convexity and smoothness of $[A_0, A_1]_\theta$ in terms of these moduli for A_0 and A_1 . In general, up to equivalence of moduli these estimates are best possible.

A result of Beauzamy [1, p. 71] states that if A_0 and A_1 are compatible Banach spaces, one of which is uniformly convex, then the spaces $S(p, \xi_0, A_0, \xi_1, A_1)$ obtained from A_0 and A_1 by the method of real interpolation, are also uniformly convex for $1 < p < \infty$. In this note we present an analogue of this result for the complex interpolation method and give an estimate for the modulus of convexity of $[A_0, A_1]_\theta$ in terms of the moduli of A_0 and A_1 . This estimate is best possible in general, up to equivalence of moduli. We remark that for the power type of the moduli, such an estimate is implicit in [10, section 1.9]. We use the notation and definitions of [7] and of [3,2] for concepts from the geometry of Banach spaces and from interpolation space theory respectively. The letters A, A_0, A_1 will always denote Banach spaces over the complex field. (A_0, A_1) is a compatible couple (interpolation pair). The complex interpolation space $[A_0, A_1]_\theta$ is denoted by A_θ .

We say that a positive function f on $\mathbf{R}^+$ is *dominated* by another function g (notation: $f < g$ or $f(t) < g(t)$) if there are positive constants a, b such that

$$f(t) \leq ag(bt) \quad \text{for all } t \in \mathbf{R}^+.$$

f is *equivalent* to g ($f \sim g$) if $f < g$ and $g < f$. We denote by f^{-1} the inverse function of f (if it exists). We denote the moduli of uniform convexity and uniform smoothness of A by $\delta_A(\epsilon)$ and $\rho_A(\tau)$ respectively.

It is known [4; 7, part II] that ρ_A is an Orlicz function and that δ_A is equivalent to an Orlicz function $\tilde{\delta}_A$ ($\delta_A(\epsilon)$ is defined only for $0 \leq \epsilon \leq 2$ but $\tilde{\delta}_A$ is defined for all $\epsilon \geq 0$ and equivalent to δ_A near 0). Also, $\rho_A(\tau) > 0$ for $\tau > 0$ (this can be concluded from the duality formula $\rho_A(\tau) = \frac{1}{2} \sup_{\epsilon \geq 0} \{\tau\epsilon - 2\delta_{A^*}(\epsilon)\}$ and the fact that $\delta_{A^*}(\epsilon) < \epsilon^2$) and being an Orlicz function, it turns out that ρ_A is strictly increasing. $\tilde{\delta}_A$ is also strictly increasing if A is uniformly convex (δ_A itself is strictly increasing when A is uniformly convex since $\delta_A(\epsilon)/\epsilon$ is nondecreasing). If L is a Banach lattice of measurable functions on a measure space (Ω, Σ, μ) we denote by $L(A)$ the space of

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A -valued Bochner measurable functions f such that $\|f\|_A \in L$, normed by $\|f\|_{L(A)} = \|\|f\|_A\|_L$. For $1 \leq p \leq \infty$ and $n \in \mathbf{N}$, we denote by l_p^n or L_p^n the space $\mathbf{R}^n$ with the norms $\|(a_i)\| = (\sum_{i=1}^n |a_i|^p)^{1/p}$ or, respectively,

$$\|(a_i)\| = \left(\frac{1}{n} \sum_{i=1}^n |a_i|^p \right)^{1/p}.$$

We conclude this introduction by recalling a construction of Calderón [3, 13.5]: If M_0 and M_1 are Banach lattices of functions on (Ω, Σ, μ) and $0 < \theta < 1$ then the Banach lattice $M_0^{1-\theta} M_1^\theta$ is defined to be the space of all measurable functions f which satisfy $|f| \leq \lambda |f_0|^{1-\theta} |f_1|^\theta$ for some $f_0 \in M_0, f_1 \in M_1, \lambda > 0$ with $\|f_i\|_{M_i} \leq 1, i = 0, 1$. The norm in $M_0^{1-\theta} M_1^\theta$ is the infimum of the values of the above λ .

Finally, we wish to thank the referee of a previous version of this paper for helpful remarks and in particular, for calling our attention to Lemma 3 which enabled us to provide a simpler proof and obtain a stronger result.

THEOREM 1. *Let (A_0, A_1) be a compatible couple of Banach spaces and $0 < \theta < 1$.*

(i) *If at least one of A_0, A_1 (e.g. A_0) is uniformly convex, then $A_\theta = [A_0, A_1]_\theta$ is uniformly convex and*

$$\delta_{A_\theta}(\varepsilon) > \delta_{A_0}(\varepsilon^{1/(1-\theta)})$$

(also, $\delta_{A_\theta}(\varepsilon) > \delta_{A_1}(\varepsilon^{1/\theta})$ if A_1 is uniformly convex).

(ii) *If both A_0 and A_1 are uniformly convex, then $\delta_{A_\theta} > \delta$ where δ is the inverse function of $(\delta_{A_0}^{-1})^{1-\theta} (\delta_{A_1}^{-1})^\theta$.*

(iii) $\rho_{A_\theta} < \rho$, where ρ is the inverse function of $(\rho_{A_0}^{-1})^{1-\theta} (\rho_{A_1}^{-1})^\theta$.

We remark that simple examples (e.g. L_p -spaces) show that the above estimates are, in general, best possible up to equivalence of moduli.

By duality between δ_A and ρ_{A^*} , δ_{A^*} and ρ_A (up to equivalence of functions), (iii) follows from (i) and (ii). For the proof of (i) and (ii) we introduce the following definitions and lemmas.

For a strictly increasing Orlicz function ϕ , we define the lattice norm $\|\cdot\|_\phi$ on $\mathbf{R}^2$ by

$$\|(a, b)\|_\phi = \inf\{\rho > 0 \mid |a|/\rho + \phi(|b|/\rho) \leq 1\},$$

which is the same as

$$\|(a, b)\|_\phi = \inf\{\lambda > 0 \mid \exists(c, d), |c| + |d| \leq 1, |a| \leq \lambda |c|, |b| \leq \lambda \phi^{-1}(|d|)\}.$$

We denote the Banach lattice $(\mathbf{R}^2, \|\cdot\|_\phi)$ by M_ϕ .

LEMMA 2. *Let ϕ_0, ϕ_1 be two strictly increasing Orlicz functions and $\phi = [(\phi_0^{-1})^{1-\theta} (\phi_1^{-1})^\theta]^{-1}$. Then*

$$(1) \quad \|\cdot\|_{\phi(\alpha \cdot)} \leq \|\cdot\|_{M_0^{1-\theta} M_1^\theta} \leq \|\cdot\|_\phi$$

where $\alpha = \min(\theta, 1 - \theta)$.

PROOF. It is clear that ϕ is an Orlicz function.

Since ϕ_i^{-1} are concave and $\phi_i^{-1}(0) = 0$ ($i = 0, 1$), we have for $0 < \lambda \leq 1$ and $t \geq 0$

$$\lambda \phi_i^{-1}(t) \leq \phi_i^{-1}(\lambda t).$$

Consequently, if $0 < t_1 \leq t_2$ we have

$$\phi_i^{-1}((1 - \theta)t_1 + \theta t_2) \geq \frac{(1 - \theta)t_1 + \theta t_2}{t_2} \phi_i^{-1}(t_2) \geq \theta \phi_i^{-1}(t_2) \geq \theta \phi_i^{-1}(t_1).$$

In a similar way, for $0 < t_2 \leq t_1$ we have

$$\phi_i^{-1}((1 - \theta)t_1 + \theta t_2) \geq (1 - \theta) \phi_i^{-1}(t_1) \geq (1 - \theta) \phi_i^{-1}(t_2),$$

hence for all $t_1, t_2 \geq 0$

$$[\phi_0^{-1}(t_1)]^{1-\theta} [\phi_1^{-1}(t_2)]^\theta \leq \alpha^{-1} [\phi_0^{-1}((1 - \theta)t_1 + \theta t_2)]^{1-\theta} [\phi_1^{-1}((1 - \theta)t_1 + \theta t_2)]^\theta$$

with $\alpha = \min(\theta, 1 - \theta)$.

Now suppose $\|(a, b)\|_{M_{\phi_0}^{1-\theta} M_{\phi_1}^\theta} \leq 1$. By the definition, there are c_1, c_2, d_1, d_2 with $|c_j| + |d_j| \leq 1$, $j = 1, 2$, such that

$$|a| \leq |c_1|^{1-\theta} |c_2|^\theta$$

and

$$|b| \leq [\phi_0^{-1}(|d_1|)]^{1-\theta} [\phi_1^{-1}(|d_2|)]^\theta$$

then

$$|a| \leq (1 - \theta) |c_1| + \theta |c_2|$$

and

$$|b| \leq \alpha^{-1} [\phi_0^{-1}((1 - \theta) |d_1| + \theta |d_2|)]^{1-\theta} [\phi_1^{-1}((1 - \theta) |d_1| + \theta |d_2|)]^\theta$$

or

$$\alpha |b| \leq \phi^{-1}((1 - \theta) |d_1| + \theta |d_2|).$$

Since

$$[(1 - \theta) |c_1| + \theta |c_2|] + [(1 - \theta) |d_1| + \theta |d_2|] \leq 1$$

this implies that $\|(a, b)\|_{\phi(\alpha \cdot)} \leq 1$ thus proving the left-hand side of (1). The right-hand side of (1) is straightforward. $\square$

LEMMA 3. $\delta_A > \delta$ if and only if there are positive constants a, b such that for all x, y in A holds

$$(2) \quad \frac{\|x + y\|^2 + \|x - y\|^2}{2} \leq 1 \Rightarrow \|x\| + a\delta(b\|y\|) \leq 1.$$

PROOF. If (2) holds, then obviously $\delta_A > \delta$ (take z, w with $\|z\|, \|w\| \leq 1$ and write $z = x + y, w = x - y$).

On the other hand, by [4], ([5]) $\delta_A \sim \delta_{L_2(A)}$. Assume $\delta_{L_2(A)} > \delta$. Given that

$$\frac{\|x + y\|^2 + \|x - y\|^2}{2} \leq 1$$

we consider x as a constant function in $L_2(A)$ over a probability space and look at the monotone basic sequence of two elements $\{x, \varepsilon y\}$ in $L_2(A)$, where ε is a Bernoulli variable. This sequence satisfies the requirement of Proposition 2.1 in [8] hence, by that proposition

$$\|x\| + \delta_{L_2(A)}(\|y\|) \leq 1. \quad \square$$

PROOF OF THEOREM 1. We assume, as we may, that δ_{A_i} are Orlicz functions, strictly increasing if A_i is uniformly convex.

We prove first (ii). By Lemma 3, (2) is satisfied by A_i, δ_{A_i} with constants a_i, b_i ($i = 0, 1$). If $\phi_i = a_i \delta_{A_i}(b_i)$ then this means that the operator

$$T: A_i \times A_i \rightarrow A_i \times A_i$$

defined by $T(z, w) = ((z + w)/2, (z - w)/2)$, has norm ≤ 1 as an operator

$$T: L_2^2(A_i) \rightarrow M_{\phi_i}(A_i)$$

for both i 's.

Hence T has norm ≤ 1 as an operator

$$T: L_2^2(A_\theta) \rightarrow [M_{\phi_0}(A_0), M_{\phi_1}(A_1)]_\theta.$$

The lattice $M_{\phi_0}^{1-\theta} M_{\phi_1}^\theta$ clearly has the convergence property required for the result in paragraph (i) of [3, 13.6] and that result yields that

$$[M_{\phi_0}(A_0), M_{\phi_1}(A_1)]_\theta = M_{\phi_0}^{1-\theta} M_{\phi_1}^\theta(A_\theta)$$

which, together with Lemma 2 yields that T has norm ≤ 1 as an operator

$$T: L_2^2(A_\theta) \rightarrow M_{\phi(\alpha)}(A_\theta)$$

where $\phi = [(\phi_0^{-1})^{1-\theta}(\phi_1^{-1})^\theta]^{-1}$.

The last assertion is just (2) for $A = A_\theta$, $\delta = \phi$, $a = 1$, $b = \alpha$ from which (ii) follows easily.

The proof of (i) goes along the same lines, we use (2) for $A_0, \delta_{A_0}, a_0, b_0$ and take into account that in every Banach space we have

$$\left(\frac{\|x + y\|^2 + \|x - y\|^2}{2} \right)^{1/2} \geq \max(\|x\|, \|y\|)$$

i.e. $T: L_2^2(A_1) \rightarrow l_\infty^2(A_1)$ has norm ≤ 1 .

Hence $T: L_2^2(A_\theta) \rightarrow [M_{\phi_0}(A_0), l_\infty^2(A_1)]_\theta$ has norm ≤ 1 ; this yields for $x, y \in A_\theta$

$$\frac{\|x + y\|^2 + \|x - y\|^2}{2} \leq 1 \Rightarrow \|x\|^{1/(1-\theta)} + \phi_0(\|y\|^{1/(1-\theta)}) \leq 1;$$

hence, if $z, w \in A_\theta$, $\|z\|, \|w\| \leq 1$ then

$$\begin{aligned} \left\| \frac{z + w}{2} \right\| &\leq \left[1 - \phi_0 \left(\left\| \frac{z - w}{2} \right\|^{1/(1-\theta)} \right) \right]^{1-\theta} \\ &\leq 1 - (1 - \theta) \phi_0 \left(\left\| \frac{z - w}{2} \right\|^{1/(1-\theta)} \right) \end{aligned}$$

which proves (i). $\square$

REMARK. Another approach which can be used to prove similar results proceeds via use of the martingale characterization of uniform convexity of [8 and 6]. A disadvantage of this approach is that in the general case it yields only isomorphic results. However, for functions $\delta(\epsilon)$ of the form ϵ^q it gives the same result as here, and in this case it works also for the real interpolation method. Namely, if A_i are isomorphic to spaces with moduli of uniform convexity $> \epsilon^{q_i}$, then $(A_0, A_1)_{\theta, s}$ is isomorphic to a space with modulus of convexity $> \epsilon^s$ for every s with $q_\theta \leq s < \infty$ ($1/q_\theta = (1 - \theta)/q_0 + \theta/q_1$).

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