STRICTLY HYPOELLIPTIC SECOND ORDER DIFFERENTIAL OPERATORS

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ABSTRACT. We shall show strict hypoellipticity of some second order differential operators which are generalized equations considered by Hörmander, Oleinik and Radkevič, using localized energy inequalities.

1. Result. Let Ω be an open set of $\mathbf{R}^n$ and $L(x, D)$ be a second order differential operator

$$(1.1) \quad L(x, D) = X_\alpha(x, D)a^{\alpha\beta}(x)X_\beta(x, D) + X_0(x, D) + b(x),$$

where $a^{\alpha\beta}(x)$, $b(x) \in C^\infty(\Omega)$, $a^{\alpha\beta}(x) = a^{\beta\alpha}(x)$ ($\alpha, \beta = 1, \dots, N$) and $X_\alpha(x, iD)$, $(\operatorname{Re} X_0)(x, iD)$ and $(\operatorname{Im} X_0)(x, iD)$ are real vector fields over Ω . In (1.1) as well as the rest of this note we use the following convention of summation:

$$\sum_{\alpha, \beta=1}^N A^{\alpha\beta} a_\alpha b_\beta = A^{\alpha\beta} a_\alpha b_\beta \quad \text{and} \quad \sum_{\alpha=1}^N A^\alpha a_\alpha = A^\alpha a_\alpha,$$

where $A^{\alpha\beta}$, A^α , a_α and b_β are functions or operators over $C^\infty(\Omega)$.

We shall define properly supported pseudo-differential operators $Y_k(x, D)$ ($k = 0, \dots, N+n$) with the following symbols:

$$Y_0(x, \xi) = (\operatorname{Im} X_0)(x, \xi), \quad Y_\alpha(x, \xi) = (\operatorname{Re} a^{\alpha\beta})(x)X_\beta(x, \xi), \\ Y_{N+j}(x, \xi) = (\partial(\operatorname{Re} a^{\alpha\beta})/\partial x_j)X_\alpha X_\beta(x, \xi) \cdot |\xi|^{-1}$$

where $\alpha = 1, \dots, N$ and $j = 1, \dots, n$. For a sequence $I = (i_1, \dots, i_s)$ ($0 \leq i_k \leq N+n$) with the length $|I| = s$, we denote the iterated commutator $[Y_{i_1}, [Y_{i_2}, \dots, [Y_{i_{s-1}}, Y_{i_s}] \dots]]$ by $Y_I(x, D)$.

We impose the following conditions on L :

(A.1) For any compact set K of Ω there exists a proper and closed convex cone $\Gamma_K \subset \{z \in \mathbf{C}; \operatorname{Re} z > 0\} \cup \{0\}$ such that

$$\{a^{\alpha\beta}(x)\eta_\alpha\eta_\beta: (x, \eta) \in K \times \mathbf{R}^N\} \subset \Gamma_K.$$

(A.2) For any compact set K of Ω there exists a constant C_K such that

$$|(\operatorname{Re} X_0)(x, \xi)|^2 \leq C_K (\operatorname{Re} a^{\alpha\beta})(x)X_\alpha X_\beta(x, \xi),$$

where $(x, \xi) \in K \times \mathbf{R}^N$.

We shall use the following notation which is a microlocalized one of Oleinik and Radkevič [3].

DEFINITION. Let Γ be an open conic subset of $T^*(\Omega) \setminus 0$. A second order differential operator $L(x, D)$ with a form (1.1) is of rank n over Γ if for any closed

Received by the editors June 12, 1981.

1980 *Mathematics Subject Classification.* Primary 35H05.

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0002-9939/81/0000-0752/\$02.25

subset of Γ' of Γ such that $\Gamma' \cap \{(x, \xi) \in T^*(\Omega) \setminus 0; |\xi| = 1\}$ is a compact set, there exist a positive integer $\alpha = \alpha(\Gamma')$ and a positive constant $C = C(\Gamma')$ such that

$$(1.3) \quad \sum_{|I| \leq \alpha} |Y_I(x, \xi)|^2 \geq C|\xi|^2$$

where $(x, \xi) \in \Gamma'$. Here $Y_I(x, \xi)$ is the principal symbol of $Y_I(x, D)$.

THEOREM. *Let $L(x, D)$ be a second order differential operator with a form (1.1), which satisfies assumptions (A.1) and (A.2). Then if $L(x, D)$ is of rank n over Γ , for any $u \in \mathcal{D}'(\Omega)$ we have*

$$\text{WF}(u) \cap \Gamma = \text{WF}(Lu) \cap \Gamma.$$

As a consequence of the above theorem, we obtain the following:

COROLLARY 1. *Second order differential operators $L(x, D)$ considered in Hörmander [1] and Oleinik and Radkevič [3] are strictly hypoelliptic, if $L(x, D)$ is of rank n over $T^*(\Omega) \setminus 0$.*

Using our theorem, we can prove the following partial hypoellipticity:

COROLLARY 2. *Let $\Omega = \Omega_1 \times \Omega_2$, where Ω_i ($i = 1, 2$) is an open set of $\mathbb{R}^{n_i}$, and $\mathcal{B}$ be a conical neighbourhood of $\xi^{(2)} = 0$ in $\mathbb{R}^{n_2}$ ($n = n_1 + n_2$). If the second order differential operator $L(x, D)$ is of rank n in Ω , then L is partially hypoelliptic on $x^{(1)}$, i.e., if $f \in C^\infty(\Omega_1, \mathcal{D}'(\Omega_2))$, $\text{WF}(f) \subset \Omega \times \mathcal{B}$ and $Lu = f$, then $u \in C^\infty(\Omega_1; \mathcal{D}'(\Omega))$.*

2. Preliminaries. First we shall state lemmas with respect to wave front set (see §2.5 of [2]).

LEMMA 2.1. *A point $(x_0, \xi_0) \in T^*(\Omega) \setminus 0$ does not belong to $\text{WF}(u)$ if and only if there exists a properly supported pseudo-differential operator $\chi(x, D)$ whose principal symbol does not vanish at (x_0, ξ_0) such that $\chi(x, D)u \in C^\infty(\Omega)$.*

LEMMA 2.2 (SEE PROPOSITION 2.5.9 OF [2]). *If $u \in \mathcal{D}'(\Omega)$, we have $A(x, D)u \in C^\infty(\Omega)$ for all properly supported pseudo-differential operators A of Ω with*

$$\text{WF}(A) \cap \text{WF}(u) = \emptyset.$$

Next we shall cite some inequalities from [4]. Let $\tilde{Y}_\alpha(x, D)$ ($\alpha = 1, \dots, N$) and $\tilde{Y}_{N+j}(x, D)$ ($j = 1, \dots, n$) be properly supported pseudo-differential operators defined by the symbols $a^{\alpha\beta}(x)X_\beta(x; \xi)$ and $(\partial a^{\alpha\beta}(x)/\partial x_j)X_\alpha X_\beta(x, \xi)|\xi|^{-1}$ respectively. Then just using the conditions (A.1) and (A.2), in [4] we proved the following:

LEMMA 2.3 (SEE LEMMA 2.3 OF [4]). (i) *There exists a properly supported pseudo-differential operator Λ_0 of order 0 such that*

$$(2.1) \quad \sum_{k=1}^{N+n} (\|Y_k u\|^2 + \|\tilde{Y}_k u\|^2) \leq C\{|\text{Re}(\Lambda_0 Lu, u)| + \|u\|^2\},$$

where $u \in C_0^\infty(K)$ and $\|\cdot\|$ and $(\cdot, \cdot)$ are the norm and the inner product of $L^2(\Omega)$, respectively.

(ii) *For any properly supported pseudo-differential operator $\Lambda(x, D)$ of order s , there exist properly supported pseudo-differential operators Λ_k ($k = 0, \dots, N+k$)*

of order s such that

$$(2.2) \quad [L, \Lambda](x, D) = \sum_{k=1}^n (\Lambda_k \tilde{Y}_k)(x, D) + \Lambda_0(x, D).$$

Let χ and χ' be properly supported pseudo-differential operators in Ω of order 0 such that the symbol of χ' is identically 1 on some neighbourhood of the support of the symbol of χ . Then we denote this relation between χ and χ' by $\chi(x, D) \subset \subset \chi'(x, D)$. From now on we assume the supports of all properly supported pseudo-differential operators of order 0 denoted by χ, χ' or χ_i are contained in a subset Γ' of Γ , which satisfies the condition of the definition of §1.

The following lemma is easily proved by (2.1) and (2.2), and Lemma 2.3.

LEMMA 2.4. *Let $\chi(x, D) \subset \subset \chi'(x, D)$. Then we have*

$$(2.3) \quad \sum_{k=1}^{n+N} (\|Y_k \chi u\|_s^2 + \|\tilde{Y}_k \chi u\|_s^2) \leq C(\|\chi' Lu\|_s^2 + \|\chi' u\|_s^2 + \|u\|_{-r}^2).$$

Here $s \in \mathbf{R}$, r is sufficiently large and $u \in C_0^\infty(K)$, where K is a compact subset of Ω .

LEMMA 2.5. *Let $\chi(x, D) \subset \subset \chi'(x, D)$ and r be a sufficiently large number. Then for any $s \in \mathbf{R}$ and sufficiently large r there exists a positive constant ϵ ,*

$$(2.4) \quad \|\chi u\|_{s+\epsilon} \leq C(\|\chi' Lu\|_s + \|\chi' u\|_s + \|u\|_{-r}),$$

where u is $C_0^\infty(K)$.

PROOF. By (1.3) and a strong Gårding inequality we have

$$(2.5) \quad \|\chi u\|_\epsilon^2 \leq C \left(\sum_{|I| \leq \alpha} \|Y_I \chi' u\|_{\epsilon-1}^2 + \|\chi' u\|^2 + \|u\|_{-r}^2 \right).$$

Using (2.3), (2.5) and the last inequality of p. 130 of [4], we can prove (2.4) as $s = 0$. By the same way as the proof of Proposition 3.3 of [4], we can prove (2.4) for an arbitrary real number s .

3. Proof of theorem. Let φ be an element of $C^\infty(\{x \in \mathbf{R}^n; |x| \leq 1\})$ with $\int \varphi(x) dx = 1$. Then for $\delta < 0$ we define the convolution operator

$$(\varphi_\delta u)(x) = \delta^{-n} \int u(y) \varphi((y-x)/\delta) dy.$$

Since φ_δ is a pseudo-differential operator with the symbol $\hat{\varphi}(\delta\xi)$, the set $\{\varphi_\delta(D); 0 \leq \delta \leq 1\}$ is a bounded set of $L^0(\mathbf{R}^n)$. And we have $\text{supp}(\varphi_\delta u) \subset \text{supp } u + \delta$ and $u(x) \in \mathcal{E}'(\Omega)$ belongs to $H_s(\Omega)$ if and only if $\|\varphi_\delta u\|_s < C$, where C does not depend on δ .

LEMMA 3.1. *Let $u(x)$ be an element of $\mathcal{E}'(\Omega)$ such that $\chi_2 u(x)$ and $(\chi_2 Lu)(x)$ belong to $H_s(\Omega)$, then $\chi_1 \tilde{Y}_k u(x)$ and $\chi_1 Y_k u(x)$ belong to $H_s(\Omega)$, if $\chi_1(x, D) \subset \subset \chi_2(x, D)$.*

PROOF. For simplicity we use the following terminology:

$$A_\delta(u) = \sum_{k=1}^{N+n} (\|\varphi_\delta \chi_1 Y_k u\|_s^2 + \|\varphi_\delta \chi_1 \tilde{Y}_k u\|_s^2)$$

and

$$B_r(u) = \|\chi_2 Lu\|_s^2 + \|\chi_2 u\|_s^2 + \|u\|_{-r}^2,$$

where r is a sufficiently large number. Let $\Lambda_s(x, D)$ be a properly supported elliptic pseudo-differential operator in Ω of order s . Then by an elliptic inequality we have

$$(3.1) \quad A_\delta(u) \leq C \sum_{k=1}^{N+n} \{(\|\Lambda_s \varphi_\delta \chi_1 \tilde{Y}_k u\|_0^2 + \|\Lambda_s \varphi_\delta \chi_1 Y_k u\|_0^2) + B_r(u)\},$$

where in this proof constant numbers do not depend on δ . Since $[Y_k, \Lambda_s \varphi_\delta \chi_1](1 - \chi_2)$ and $[\tilde{Y}_k, \Lambda_s \varphi_\delta \chi_1](1 - \chi_2)$ belong to $L^{-\infty}(\Omega)$, by (2.1) and (3.1), we have

$$(3.2) \quad A_\delta(u) \leq C(|(\Lambda_0 L(\Lambda_s \varphi_\delta \chi_1 u), \Lambda_s \varphi_\delta \chi_1 u)| + B_r(u)).$$

Using (2.2) and (3.2), there exist properly supported pseudo-differential operators $\Lambda_{s,j}(x, D)$ of order s such that

$$(3.3) \quad A_\delta(u) \leq C \left(\sum_{j=1}^{N+n} |(\Lambda_{s,j} \chi_2 u, \tilde{Y}_k^* \Lambda_s \varphi_\delta \chi_1 u)| + B_r(u) \right).$$

Since by the definitions of Y_k and $\tilde{Y}_k^* = Y_k - i(\tilde{Y}_k - Y_k) + \tilde{Y}_0$, where $\tilde{Y}_0(x, D)$ is of order 0, by (3.3) we have $A_\delta(u) \leq C B_r(u)$, using $xy \leq \epsilon x^2 + y^2/\epsilon$ for any $\epsilon > 0$ and nonnegative numbers x and y . This completes the proof of Lemma 3.1.

LEMMA 3.2. Let $u(x)$ be an element of $\mathcal{E}'(\Omega)$ such that $\chi_2 u(x)$, $\chi_2 Lu \in H_s(\Omega)$, then there exists a positive number ϵ such that $\chi_1 u \in H_{s+\epsilon}(\Omega)$ if $\chi_1(x, D) \subset \subset \chi_2(x, D)$.

PROOF. We shall use the same terminology as that of Lemma 3.1. Since if $u \in \mathcal{E}'(K)$, where K is a compact set of Ω , there exists a compact subset K_1 of Ω such that $\varphi_\delta u \in C_0^\infty(K_1)$, we can use (2.4). Thus

$$\|\varphi_\delta \chi_1 u\|_{s+\epsilon}^2 \leq C \{ \| [L, \varphi_\delta \chi'] u \|_s^2 + \| [\chi', L] \varphi_\delta u \|_s^2 + B_r(u) \},$$

where C does not depend on δ and $\chi'(x, D)$ is a properly supported pseudo-differential operator in Ω of order 0 such that $\chi_1 \subset \subset \chi' \subset \subset \chi_2$. Then since $[L, \varphi_\delta \chi_2] \in L^{-\infty}(\Omega)$ and $\chi' Y_k u \in H_s(\Omega)$ ($k = 1, \dots, N+n$) by Lemma 3.1, we have $\|\varphi_\delta \chi_1 u\|_{s+\epsilon} < C$, where C does not depend on δ . This completes the proof of Lemma 3.2.

Let us prove the theorem. Since it is well known $\text{WF}(Lu) \subset \text{WF}(u)$ for any pseudo-differential operator L of Ω and $u \in \mathcal{D}'(\Omega)$, we shall prove $\text{WF}(u) \subset \text{WF}(Lu)$, i.e., if $(x_0, \xi_0) \notin \text{WF}(Lu)$, then $(x_0, \xi_0) \notin \text{WF}(u)$. Assume $(x_0, \xi_0) \in T^*(\Omega) \setminus 0$ does not belong to $\text{WF}(Lu)$. Then using a cut-off function which is not zero at x_0 , we may assume $u \in \mathcal{E}'(\Omega)$ and $(x_0, \xi_0) \notin \text{WF}(Lu)$. By Lemma 2.1 if we show a properly supported pseudo-differential operator $\chi(x, D)$ in Ω of order 0 such that $\chi u \in C^\infty(\Omega)$ and χ is elliptic near (x_0, ξ_0) , then $(x_0, \xi_0) \notin \text{WF}(u)$. Here we assume $(x_0, \xi_0) \in \Gamma$.

Since $(x_0, \xi_0) \notin \text{WF}(Lu)$, by Lemma 2.2, there exists an open conic neighbourhood $\bar{\Gamma}$ of (x_0, ξ_0) such that $\chi_0(x, D)Lu \in C^\infty(\Omega)$ if the support of the symbol of χ_0 is contained in $\bar{\Gamma}$. Let Γ' be an open conic subset of Γ such that for some $\alpha > 0$, $\Gamma_1 = \{(x, \xi); d((x, \xi), \Gamma) = \inf(|x - x'| + |\xi|/|\xi'| - \xi'/|\xi'|) < \alpha; (x', \xi') \in \Gamma'\} \subset \bar{\Gamma}$. Then we define $\Gamma_n = \{(x, \xi); d((x, \xi), \Gamma') < \alpha/n\}$ and $\chi_n(x, D)$ is a properly supported pseudo-differential operator such that the support of the symbol of χh_n ($n > 2$) is contained in Γ_{2n-1} and the symbol is equal to 1 on Γ_{2n} . Since $\chi_1 \supset \supset \chi_2 \supset \supset \dots$, inductively applying Lemma 3.2 to χ_{n-1} , χ_n , u as $\chi_1 = \chi_n$ and $\chi_2 = \chi_{n-1}$, we have $\chi_n u \in H_{s+(n-1)\epsilon}(\Omega)$. If the support of the symbol of a properly supported pseudo-differential operator is contained in Γ' then $\chi u \in H_{s+n\epsilon}(\Omega)$ for any positive integer n . This means $\chi u \in C^\infty(\Omega)$. Thus if $\chi(x, D)$ is elliptic in (x_0, ξ_0) , we have $(x_0, \xi_0) \notin \text{WF}(u)$. This completes the proof of the theorem.

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