A UNIQUENESS RESULT IN CONFORMAL MAPPING. II

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ABSTRACT. This paper gives an elementary proof of the result that for a function f in the family Σ the diameter of the complement of the image of $|z| > 1$ by $w = f(z)$ attains its minimal value 2 only for $f(z) = z + c$, c constant.

1. Some years ago the author gave the first published proof [2] of the result that if, for a function f in the family Σ [1], the diameter of the complement of the image of $|z| > 1$ by $w = f(z)$ attains its minimal value 2, then $f(z) = z + c$ (c constant). While the proof is very simple, it utilizes results and concepts which would not normally be familiar to a student in a good basic course in Function Theory, which is the natural context for the above result. More recently Pfluger [3, 4] has given several versions of a proof of this result which seem to him to be more elementary. It appears that they also would require some digression from the material usually found in a Function Theory course. Our purpose here is to give an elementary proof in purely Function Theoretic terms.

2. LEMMA 1. *Let $f \in \Sigma$ have Laurent expansion about the point at infinity*

$$(1) \quad f(z) = z + c_0 + \sum_{n=1}^{\infty} c_n z^{-n}.$$

Let the complement of the image of $|z| > 1$ under $w = f(z)$ have diameter D . Then $D \geq 2$ and a necessary condition for equality is $c_{2n+1} = 0$, n integral ≥ 0 .

Let D_r be the diameter of the image of $|z| = r$, $r > 1$, under $w = f(z)$; $D_r^* = \max_{|z|=r} |f(z) - f(-z)|$. Evidently $D_r \geq D_r^* \geq D$ and $\lim_{r \rightarrow 1} D_r = D$. Further

$$(D_r^*)^2 \geq \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta}) - f(-re^{i\theta})|^2 d\theta = 4r^2 + 4 \sum_{n=0}^{\infty} |c_{2n+1}|^2 r^{-2(2n+1)}.$$

Thus $D \geq 2$ with equality only if $c_{2n+1} = 0$, $n = 0, 1, 2, \dots$

The following lemma is found in [3].

LEMMA 2. *Using the same notation as in Lemma 1, $D_r \leq rD$, $r > 1$.*

Let $|z_1| = |z_2| = r$, $r > 1$, be such that $|f(z_1) - f(z_2)| = D_r$ and let $\Phi(z) = f(z_1 z_2^{-1} z) - f(z)$. Then for $\rho > 1$, $\max_{|z|=\rho} |\Phi(z)| \leq D_\rho$. The function $z^{-1}\Phi(z)$ is regular outside the unit circle (including the point at infinity) thus $|z^{-1}\Phi(z)| \leq \rho^{-1}D_\rho$ in $|z| \geq \rho$. Letting ρ tend to 1 we have $|z^{-1}\Phi(z)| \leq D$ in $|z| > 1$. Thus $D_r \leq rD$.

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THEOREM. *Let $f \in \Sigma$ and let the complement of the image of $|z| > 1$ under $w = f(z)$ have diameter D . Then $D \geq 2$ and $D = 2$ if and only if $f(z) = z + c$, c constant.*

It remains only to consider the possibility $D = 2$. If $f(z)$ is not $z + c$ let it have the Laurent expansion (1). By Lemma 1 the first nonzero coefficient beyond the constant term would be c_{2n} , $n \geq 1$. By Lemmas 1 and 2, $D_r = D_r^* = 2r$ thus $|f(re^{i\theta}) - f(re^{i\varphi})|^2$ would be maximized for $\varphi = \theta + \pi$ for all $r > 1$, θ real and the partial derivative of this quantity with respect to φ evaluated at $\varphi = \theta + \pi$ would be identically zero in this set. Inserting the expansion (1) for f the coefficient of r^{-2n+1} would be

$$4ni(c_{2n}e^{-(2n+1)i\theta} - \bar{c}_{2n}e^{(2n+1)i\theta})$$

which would imply the contradiction $c_{2n} = 0$.

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