AN INFINITELY DIVISIBLE DISTRIBUTION INVOLVING MODIFIED BESSEL FUNCTIONS

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ABSTRACT. We prove that the function

$$\left(\frac{b}{a}\right)^{\mu-\nu} \frac{K_{\mu}(bx^{1/2})K_{\nu}(ax^{1/2})}{K_{\mu}(ax^{1/2})K_{\nu}(bx^{1/2})}$$

is the Laplace transform of an infinitely divisible probability distribution when $\nu > \mu \geq 0$ and $b > a > 0$. This implies the complete monotonicity of the function. We also establish a representation as a Stieltjes transform, which implies in particular that the function has positive real part when x lies in the right half-plane. We conjecture that

$$\left(\frac{b}{a}\right)^{\mu-\nu} \frac{I_{\mu}(ax^{1/2})I_{\nu}(bx^{1/2})}{I_{\mu}(bx^{1/2})I_{\nu}(ax^{1/2})}$$

also is the Laplace transform of an infinitely divisible probability distribution. It is known that in the limit as $\nu \rightarrow \infty$, the infinite divisibility property holds for both functions.

1. Introduction. A probability distribution on $[0, \infty)$ is said to be infinitely divisible if and only if for every natural number n , the n th root of its Laplace transform is a Laplace transform of a probability distribution. A function f defined on $(0, \infty)$ and of class C^{∞} is said to be completely monotonic if $(-1)^n f^{(n)}(x) \geq 0$ on $(0, \infty)$ for all n . The connection between infinitely divisible distributions and completely monotonic functions is expressed in the following theorem (see [5, p. 450]):

THEOREM 1. *The function w is the Laplace transform of an infinitely divisible probability distribution on $[0, \infty)$ if and only if $w = e^{-h}$ where $h(0+) = 0$ and h' is completely monotonic.*

In §2 we shall prove the following result:

THEOREM 2. *Let K_{λ} be the modified Bessel function of the second kind. Then*

$$(1.1) \quad F(x; \mu, \nu) = \left(\frac{b}{a}\right)^{\mu-\nu} \frac{K_{\mu}(bx^{1/2})K_{\nu}(ax^{1/2})}{K_{\mu}(ax^{1/2})K_{\nu}(bx^{1/2})}$$

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is the Laplace transform of an infinitely divisible probability distribution when $\nu > \mu \geq 0$ and $b > a > 0$.

As ν increases without limit,

$$K_\nu(x) \sim (2\nu)^{\nu-1/2} e^{-\nu} x^{-\nu} \pi^{1/2}$$

(see, for example [8]). Hence

$$F(x; \mu, \infty) = \left(\frac{b}{a}\right)^\mu \frac{K_\mu(bx^{1/2})}{K_\mu(ax^{1/2})}.$$

The limiting case ($\nu \rightarrow \infty$) of Theorem 2 was proved by Ismail and Kelker [9] using complex variable and special functions methods. Kent [11] and Wendel [14] independently encountered the same distribution ($\nu = \infty$) in their studies of Bessel processes and Brownian motion, and established its infinite divisibility property by probabilistic methods. Ismail and May [10] provided an integral representation of the probability density function as the Laplace transform of a transcendental function. Miller [12, p. 82] proved that $F(x; \mu, \nu)$ is monotone decreasing, a result needed in certain hypothesis testing problems.

In §3 we shall establish an integral representation of $F(x; \mu, \nu)$ as a Stieltjes transform. This implies, for example, that $F(z; \mu, \nu)$ has positive real part when z lies in the right half-plane. The Stieltjes transform seems to play an important role in infinite divisibility problems, and in the theory of orthogonal polynomials (see [2, 3, 9, 10]).

We conjecture that if $\nu > \mu \geq 0$ and $b > a > 0$, then the function

$$(1.2) \quad G(x; \mu, \nu) = \left(\frac{b}{a}\right)^{\mu-\nu} \frac{I_\mu(ax^{1/2})I_\nu(bx^{1/2})}{I_\mu(bx^{1/2})I_\nu(ax^{1/2})}$$

is also the Laplace transform of an infinitely divisible probability distribution; here I_λ is the modified Bessel function of the first kind.

As λ increases without limit

$$I_\lambda(x) \sim (\tfrac{1}{2}x)^\lambda \Gamma^{-1}(\lambda + 1)$$

(see, for example [4, pp. 22 and 23]). Hence

$$G(x; \mu, \infty) = \left(\frac{b}{a}\right)^\mu \frac{I_\mu(ax^{1/2})}{I_\mu(bx^{1/2})}$$

and the conjecture reduces to a known result, see [9, 10, 14]. In §4 we shall discuss this conjecture and formulate an equivalent conjecture involving functions that resemble theta functions.

2. Proof of Theorem 2. Our argument will be to show that

$$(2.1) \quad h(x) = -\ln F(x; \mu, \nu)$$

has a completely monotonic derivative and that $h(0+) = 0$. Then we may invoke Theorem 1 to complete the proof.

Clearly h is well defined since $F(x; \mu, \nu)$ is positive on $[0, \infty)$. Now (see [1, p. 375]) as $x \rightarrow 0+$,

$$\begin{aligned} K_\nu(x) &\sim \tfrac{1}{2}\Gamma(\nu)(\tfrac{1}{2}x)^{-\nu}, \quad \operatorname{Re} \nu > 0, \\ K_0(x) &\sim -\ln x. \end{aligned}$$

This implies that $h(0+) = 0$ since $F(0+; \mu, \nu) = 1$. Using the recursion relation [13, p.79]

$$(2.2) \quad K'_\nu(x) = -K_{\nu-1}(x) - \nu K_\nu(x)/x$$

we obtain

$$(2.3) \quad \begin{aligned} -2x^{1/2}h'(x) &= a \frac{K_{\mu-1}(ax^{1/2})}{K_\mu(ax^{1/2})} - b \frac{K_{\mu-1}(bx^{1/2})}{K_\mu(bx^{1/2})} \\ &\quad - a \frac{K_{\nu-1}(ax^{1/2})}{K_\nu(ax^{1/2})} + b \frac{K_{\nu-1}(bx^{1/2})}{K_\nu(bx^{1/2})}. \end{aligned}$$

The integral representation [7]

$$(2.4) \quad \frac{K_{\nu-1}(x^{1/2})}{K_\nu(x^{1/2})} = \frac{2}{\pi^2} \int_0^\infty \frac{x^{1/2}t^{-1}}{x+t} [J_\nu^2(t^{1/2}) + Y_\nu^2(t^{1/2})]^{-1} dt, \quad \nu \geq 0,$$

where J_ν and Y_ν are the Bessel functions of the first and second kind, respectively, enables us to express h' in the form

$$(2.5) \quad h'(x) = \frac{b^2 - a^2}{\pi^2} \int_0^\infty (xa^2 + t)^{-1}(xb^2 + t)^{-1} \phi(t) dt$$

where

$$\phi(t^2) = [J_\mu^2(t) + Y_\mu^2(t)]^{-1} - [J_\nu^2(t) + Y_\nu^2(t)]^{-1}.$$

Now, Nicholson's formula [13, p. 444]

$$(2.6) \quad J_\lambda^2(x) + Y_\lambda^2(x) = \frac{8}{\pi^2} \int_0^\infty K_0(2x \sinh t) \cosh 2\lambda t dt$$

shows that $J_\lambda^2(x) + Y_\lambda^2(x)$ is a strictly increasing function of λ for $\lambda \geq 0$ and x a fixed positive number. Therefore $\phi(t)$ is strictly positive for $t > 0$; and the integrand in equation (2.5) is a positive multiple of the completely monotonic function $(t + a^2x)^{-1}(t + b^2x)^{-1}$ (as a function of x). This then implies the complete monotonicity of h' . Thus the function $F(x; \mu, \nu) = e^{-h(x)}$ is also completely monotonic (see Criterion 2 in [5, p. 441]). By Bernstein's theorem [5, p. 439] we see that $F(x; \mu, \nu)$ must be the Laplace transform of a probability distribution since $F(0; \mu, \nu) = 1$.

Note that in the process of proving Theorem 2 we also have shown that:

COROLLARY. *The function $F(x; \mu, \nu)$, defined in (1.1), is a completely monotonic function of x for $\nu \geq \mu \geq 0$ and $b > a > 0$.*

3. An integral representation. The purpose of this section is to establish an integral representation for the function F of Theorem 2.

THEOREM 3. *If K_λ is the modified Bessel function of the second kind and of order λ , then*

$$(3.1) \quad \begin{aligned} &\frac{K_\mu(bz^{1/2})K_\nu(az^{1/2})}{K_\mu(az^{1/2})K_\nu(bz^{1/2})} - 1 \\ &= \frac{1}{\pi} \int_0^\infty \frac{M(t, a, b, \mu, \nu) dt}{(z+t)[J_\mu^2(at^{1/2}) + Y_\mu^2(at^{1/2})][J_\nu^2(bt^{1/2}) + Y_\nu^2(bt^{1/2})]}, \end{aligned}$$

$|\arg z| < \pi,$

for $\mu > 0, \nu > 0, a > 0, b > 0$ where

$$(3.2) \quad \begin{aligned} M(t^2, a, b, \mu, \nu) = & [J_\mu(at)J_\nu(at) + Y_\mu(at)Y_\nu(at)] \\ & \times [J_\nu(bt)Y_\mu(bt) - J_\mu(bt)Y_\nu(bt)] \\ & - [J_\mu(bt)J_\nu(bt) + Y_\mu(bt)Y_\nu(bt)] \\ & \times [J_\nu(at)Y_\mu(at) - J_\mu(at)Y_\nu(at)] \end{aligned}$$

and J_μ, Y_ν are the Bessel functions of the first and second kind respectively.

The proof utilizes the representation and inversion theorems for the Stieltjes transform, namely,

LEMMA (REPRESENTATION THEOREM). If

- (i) $H(z)$ is analytic for $|\arg z| < \pi/\alpha$ for some $\alpha, 0 < \alpha < 1$, and
- (ii) $H(z) = o(1)$ as $|z| \rightarrow \infty$ and $H(z) = o(|z|^{-1})$ as $|z| \rightarrow 0$ uniformly in every sector $|\arg z| < \pi/\alpha'$ with $\alpha' > \alpha$, then

$$(3.3) \quad H(z) = \frac{1}{\pi} \int_0^\infty \frac{dt}{x+t} \frac{1}{2\pi i} \int_C \frac{\zeta H(t\zeta) e^{\zeta/2}}{\zeta^2 + \pi^2} d\zeta$$

where C is a rectifiable closed curve going around $[-i\pi, i\pi]$ in the positive direction and lying in the strip $|\operatorname{Im} \zeta| < \pi/\alpha$.

For a proof of this lemma see [6, pp. 210 and 235]. We are now in a position to prove our theorem.

PROOF OF THEOREM 3. Denote the left-hand side of (3.1) by $H(z)$. Then $H(z)$ satisfies condition (i) of the Lemma, for some $\alpha > 2/3$, because $K_\lambda(z)$ has finitely many zeros and none in the half-plane $\operatorname{Re} z \geq 0$. For $\lambda \geq 0$ we have $K_\lambda(z) \sim (\pi/2z)^{1/2} e^{-z}$ uniformly as $|z| \rightarrow \infty$ in the sector $|\arg z| \leq 3/2 - \delta$ for any $\delta \in (0, 3\pi/2)$, (see [13, p. 203]). As $|z| \rightarrow 0$ we have $K_\lambda(z) \sim \frac{1}{2}\Gamma(\lambda)(\frac{1}{2}z)^{-\lambda}$ for $\lambda > 0$ and $K_0(z) \sim -\ln z$, (see [1, p. 375]). Hence $H(z)$ also satisfies condition (ii) of the Lemma.

The contour integral in (3.3) is now readily seen to be

$$(3.4) \quad \frac{i}{2} [H(te^{i\pi}) - H(te^{-i\pi})].$$

Equation (3.1) now follows, for $z > 0$, from (3.4) and

$$K_\nu(ze^{i\pi/2}) = -\frac{1}{2}i\pi e^{-i\pi\nu/2} [J_\nu(z) - iY_\nu(z)]$$

and

$$K_\nu(ze^{-i\pi/2}) = \frac{1}{2}i\pi e^{i\pi\nu/2} [J_\nu(z) + iY_\nu(z)],$$

(see, for example, [4, pp. 4 and 6]). Finally, (3.1) follows for $|\arg z| < \pi$ from $z > 0$ by analytic continuation.

4. Discussion of conjecture. We shall show that the conjecture made in §1 is equivalent to the statement that the function

$$(4.1) \quad \Theta_\nu(x; a, b) = \sum_{n=1}^{\infty} (e^{-j_{\nu,n}^2 ax} - e^{-j_{\nu,n}^2 bx}), \quad 0 < a < b, \quad \nu \geq 0,$$

is a decreasing function of ν , where $j_{\nu,1} < j_{\nu,2} < \dots < j_{\nu,n} < \dots$ are the successive positive zeros of the Bessel function of the first kind and order ν .

From (1.2)

$$(4.2) \quad -2x^{1/2} \frac{d}{dx} \ln G(x; \mu, \nu) = b \frac{I_{\mu+1}(bx^{1/2})}{I_{\mu}(bx^{1/2})} - a \frac{I_{\mu+1}(ax^{1/2})}{I_{\mu}(ax^{1/2})} \\ - b \frac{I_{\nu+1}(bx^{1/2})}{I_{\nu}(bx^{1/2})} + a \frac{I_{\nu+1}(ax^{1/2})}{I_{\nu}(ax^{1/2})}$$

where we have used [13, p. 79]

$$I'_{\lambda}(x) = I_{\lambda+1}(x) + \lambda I_{\lambda}(x)/x.$$

An analog of (2.4) is the Mittag-Leffler expansion [4, p. 61]

$$(4.3) \quad \frac{I_{\lambda+1}(z)}{I_{\lambda}(z)} = 2z \sum_{n=1}^{\infty} (z^2 + j_{\lambda,n}^2)^{-1}, \quad \lambda > -1.$$

Combining (4.1), (4.2) and (4.3) we get

$$-\frac{d}{dx} G(x; \mu, \nu) = \sum_{n=1}^{\infty} [(x + j_{\mu,n}^2 b^{-2})^{-1} - (x + j_{\mu,n}^2 a^{-2})^{-1}] \\ - \sum_{n=1}^{\infty} [(x + j_{\nu,n}^2 b^{-2})^{-1} - (x + j_{\nu,n}^2 a^{-2})^{-1}] \\ = \int_0^{\infty} e^{-xt} [\Theta_{\mu}(t; b^{-2}, a^{-2}) - \Theta_{\nu}(t; b^{-2}, a^{-2})] dt$$

which shows that the conjecture is equivalent to the above-mentioned property of $\Theta_{\nu}(x; a, b)$.

The sums appearing in (4.1) resemble theta functions, and, in fact, when $\nu = \pm \frac{1}{2}$ the function Θ_{ν} reduces to the difference of two theta functions,

$$\Theta_{1/2}(x; a, b) = \frac{1}{2} [\vartheta_3(0, q) - \vartheta_3(0, q')], \\ \Theta_{-1/2}(x; a, b) = \frac{1}{2} [\vartheta_2(0, q) - \vartheta_2(0, q')],$$

where $q = e^{-\pi^2 ax}$ and $q' = e^{-\pi^2 bx}$.

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