A CHARACTERIZATION OF THE MULTIPARAMETER WIENER PROCESS AND AN APPLICATION

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ABSTRACT. In this note we give an application of a characterization of the multiparameter Wiener process and of the invariance principle for R^k -valued martingales.

1. Introduction. Let q be a positive integer, $I^q = [0, 1]^q$, the points in I^q will be denoted by $\mathbf{t}, \mathbf{s}, (t_1, t_2, \dots, t_q)$, etc. I^q is partially ordered by stipulating $\mathbf{t} \leq \mathbf{s}$ if and only if $t_i \leq s_i$ for $i = 1, \dots, q$. A block B in I^q is a subset of the form $B = (\mathbf{s}, \mathbf{t}] = \prod_{j=1}^q (s_j, t_j]$ for $0 \leq \mathbf{s} \leq \mathbf{t} \leq 1$. For a function f defined on I^q we define the increment of f around the block B by

$$(1.1) \quad f(B) = \sum_{\substack{\epsilon_j = 0, 1 \\ j=1, \dots, q}} (-1)^{q-\sum \epsilon_j} f(s_1 + \epsilon_1(t_1 - s_1), \dots, s_q + \epsilon_q(t_q - s_q)).$$

Similarly if $C = (\mathbf{u}, \mathbf{v}]$ is a block of I^{q-1} we define for $i = 1, \dots, q$ and $0 \leq t \leq 1$

$$(1.2) \quad \begin{aligned} f^{(i)}(t; C) = & \sum_{\substack{\epsilon_j = 0, 1 \\ j=1, \dots, q-1}} (-1)^{q-1-\sum \epsilon_j} \\ & \cdot f(u_1 + \epsilon_1(v_1 - u_1), \dots, u_{i-1} + \epsilon_{i-1}(v_{i-1} - u_{i-1}), \\ & t, u_i + \epsilon_i(v_i - u_i), \dots, u_{q-1} + \epsilon_{q-1}(v_{q-1} - u_{q-1})). \end{aligned}$$

Notice

$$f^{(i)}(t; C) = f(((u_1, \dots, u_{i-1}, 0, u_i, \dots, u_{q-1}), (v_i, \dots, v_{i-1}, t, v_i, \dots, v_{q-1})))$$

if $f(\mathbf{s}) = 0$ whenever $s_i = 0$.

Let $C(I^q)$ be the set of all continuous functions on I^q with the uniform metric and let $D^q = D(I^q)$ be the Skorohod function space on I^q . See Bickel-Wichura [1] for further details.

The Wiener process $\{W(\mathbf{t}), \mathbf{t} \in I^q\}$ on D^q is characterized by

$$(1.3) \quad \begin{aligned} (a) \quad & P(W \in C(I^q)) = 1, \\ (b) \quad & W(\mathbf{t}) \text{ vanishes at the lower boundary of } I^q, \end{aligned}$$

that is, $P(W(\mathbf{t}) = 0) = 1$ if $t_i = 0$ for some i .

(c) If $B_1, B_2, \dots, B_k$ are pairwise disjoint blocks of I^q then the increments $W(B_1), \dots, W(B_k)$ are independent normal random variables with mean zero and variances $\lambda_q(B_1), \dots, \lambda_q(B_k)$, λ_q being the Lebesgue measure on I^q .

Lemma 1 below gives a characterization of the q -parameter Wiener process in terms of k -dimensional Wiener processes. Since its proof is straightforward it will be omitted.

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LEMMA 1. Let $\{X(t), t \in I^q\}$ be a process that vanishes at the lower boundary of I^q and has sample paths in D^q with $P(X \in C^q) = 1$. Then $X(t)$ is the Wiener process on D^q if and only if for some i ($i = 1, \dots, q$) the process $(X^{(i)}(t; C_1), \dots, X^{(i)}(t; C_k))$ is a k -dimensional Wiener process with the appropriate variance, where k is an arbitrary positive integer and the C_j 's are arbitrary disjoint blocks of I^{q-1} .

2. A convergence theorem. Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\{\mathcal{F}_t, t \in I^q\}$ be a family of σ -subfields such that $\mathcal{F}_s \subset \mathcal{F}_t$ if $s \leq t$. For $0 \leq t \leq 1$ we define for $i = 1, \dots, q$

$$(2.1) \quad \mathcal{F}_t^{(i)} = \mathcal{F}_{(1, \dots, t, 1, \dots, 1)} \text{ with } t \text{ on the } i\text{th position.}$$

We shall consider random fields $\{X(t), t \in I^q\}$ adapted to a nondecreasing family of σ -fields $\{\mathcal{F}_t^X\}$ that vanishes at the lower boundaries of I^q . By taking $\mathcal{F}_t = \mathcal{F}_t^X \times \mathcal{F}_t$ for $t \in I^q$ we may consider $X(t)$ and $W(t)$ adapted to the same family of σ -fields $\{\mathcal{F}_t\}$, where $\mathcal{F}_t = \sigma(W(s), s \leq t)$.

THEOREM 1. Let $\{X_n(t), t \in I^q\}_{n \geq 1}$ be a sequence of processes adapted to $\mathcal{F}_t$ with sample paths in D^q . Fix i . Suppose that for $k = 1, 2, \dots$ and $C_1, \dots, C_k$ disjoint blocks of I^{q-1} we have

$$(2.2) \quad (X_n^{(i)}(s^1; C_1), \dots, X_n^{(i)}(s^k; C_k)) \xrightarrow{D} (W^{(i)}(s^1; C_1), \dots, W^{(i)}(s^k; C_k))$$

for $s^1, s^2, \dots, s^k \in I$, where $\xrightarrow{D}$ stands for convergence in distribution.

Then the finite dimensional distributions of X_n converge to those of W .

PROOF. (i) For $t \in I^q$ we have $X_n(t) \xrightarrow{D} W(t)$ since $X_n(t) = X_n^{(i)}(t; C)$ where $C = \prod_{j \neq i} (0, t_j]$.

(ii) To see that $(X_n(t^1), \dots, X_n(t^k)) \xrightarrow{D} (W(t^1), \dots, W(t^k))$ for $t^1, \dots, t^k \in I^q$, write $X_n(t^i) = X_n^{(i)}(t_i^1; C_i)$ and $W(t^i) = W^{(i)}(t_i^1; C_i)$ with $C_i = \prod_{j \neq i} (0, t_j^1]$. The I^{q-1} blocks $C_1, \dots, C_k$ are not necessarily disjoint, but given $a_1, a_2, \dots, a_k \in R$ we can write $\sum_{i=1}^k a_i X_n^{(i)}(t_i^1; C_i)$ as a linear combination of random variables of the form $X_n^{(i)}(t_i^p; E_p)$ where the E_p 's are disjoint blocks of I^{q-1} . $\square$

3. Invariance principle for R^k -valued martingale. Theorem 2 below is just a k -dimensional version of Billingsley's theorem [2]. Although we could prove it by using either Billingsley's arguments or McLeish's arguments [8], it can be done by Trotter's operator method [10] as outlined below.

THEOREM 2. Let $Y_1, Y_2, \dots$ be a sequence of R^k -valued random vectors, $Y_n = (Y_n(1), \dots, Y_n(k))$, stationary, ergodic and such that

$$(3.1) \quad \begin{aligned} E(Y_n^2(j)) &= \alpha_j, \quad \text{positive and finite,} \\ E(Y_n(i)Y_n(j)) &= 0, \quad i \neq j, \\ E(Y_n | Y_1, \dots, Y_{n-1}) &= 0. \end{aligned}$$

Then $S_n = \sum_{i=1}^n Y_i / \sqrt{n}$ converges in distribution to Z , where Z has the multivariate normal distribution with mean zero, $EZ^2(j) = \alpha_j$ and $EZ(i)Z(j) = 0$ for $i \neq j$.

PROOF. To avoid messy notation let $k = 2$. For a random vector (V, U) define

$$T_{V,U}f(x)g(y) = \iint f(v+x)g(u+y)dF_{V,U}(x,y).$$

Let $C_0^2 = \{f: R \rightarrow R \text{ such that } f, f', f'' \text{ are bounded and uniformly continuous}\}$. To show that $(S_n(1), S_n(2)) \xrightarrow{D} (Z(1), Z(2))$ it is enough to verify that for all $f, g \in C_0^2$ we have

$$(3.2) \quad \lim_{n \rightarrow \infty} |T_{S_n(1), S_n(2)}fg - T_{Z(1), Z(2)}fg| = 0.$$

Write

$$(T_{S_n} - T_Z)fg = (A + B)fg$$

where

$$B = T_{U_1} \cdots T_{U_n} - T_Z$$

and

$$A = \sum_{j=0}^{n-2} T_{U_{n-j+1}} \cdots T_{U_{n+1}} [T_{V_{n-j}} - T_{U_{n-j}} T_{V_{n-j-1}}]$$

with $U_j(i) = Y_j(i)/\sqrt{n}$, $U_{n+1} = (0, 0)$ and $V_j(i) = \sum_{l=1}^j U_l(i)$. Since

$$T_{V_{n-j}}f(x)g(y) = \int f(a_1 + \cdots + a_{n-j} + x) \cdot g(b_1 + \cdots + b_{n-j} + y) dF_{U_1, \dots, U_{n-j}}(a_1, b_1, \dots, a_{n-j}, b_{n-j})$$

we expand $f(x)g(y)$ in Taylor series around $(a_1 + \cdots + a_{n-j-1} + x, b_1 + \cdots + b_{n-j-1} + y)$ and

$$\begin{aligned} T_{V_{n-j}}f(x)g(y) &= E(f(\cdot)g(\cdot) + U_{n-j}(1)f'(\cdot)g(\cdot) \\ &\quad + U_{n-j}(2)f(\cdot)g'(\cdot) + U_{n-j}(1)U_{n-j}(2)f'(\cdot)g'(\cdot) \\ &\quad + \frac{1}{2}U_{n-j}^2(1)f''(\cdot)g(\cdot) + \frac{1}{2}U_{n-j}^2(2)f(\cdot)g''(\cdot) + R_j) \end{aligned}$$

where $f(\cdot) = f(V_{n-j}(1) + x)$ and $g(\cdot) = g(V_{n-j}(2) + y)$. The term R_j can be estimated as in [10], hence given $\epsilon > 0$ for n sufficiently large we have $E|R_j| \leq \epsilon n^{-1}$. Making a similar estimate for $T_{U_{n-j}}T_{V_{n-j-1}}$ and using condition (3.1) we have

$$\begin{aligned} (T_{V_{n-j}} - T_{U_{n-j}}T_{V_{n-j-1}})fg &= E \left(R_j + R'_j + \frac{1}{2}f''g \left[E(U_{n-j}^2(1)|G_{n-j-1}) - \frac{\alpha_1}{n} \right] \right. \\ &\quad \left. + \frac{1}{2}fg'' \left[E(U_{n-j}^2(2)|G_{n-j-1}) - \frac{\alpha_2}{n} \right] \right) \end{aligned}$$

where $G_j = \sigma(Y_1, \dots, Y_j)$.

With similar estimates for all the remaining terms we have

$$\begin{aligned} |(T_{S_n} - T_Z)fg| &\leq 2\epsilon + \frac{1}{2}|f''| |g| E \left(\left| \sum_{i=1}^n U_i^2(1) - \alpha_1 \right| \right) \\ &\quad + \frac{1}{2}|f| |g''| E \left(\left| \sum_{i=1}^n U_i^2(2) - \alpha_2 \right| \right) \end{aligned}$$

where $|\cdot|$ is the supremum norm. The proof is completed by using the ergodicity. $\square$

4. Application. In this section we shall consider a random net $\{\xi_n, n \in Z^2\}$, stationary with $E\xi_n^2 = \sigma^2 > 0$ and finite, where Z stands for the integers. Let

$$\begin{aligned}\mathcal{F}_m &= \sigma(\xi_j, j \leq m), \quad \mathcal{F}_t = \prod_m \mathcal{F}([m_1 t_1], [m_2 t_2]), \\ S_{n_1 0} &= S_{0 n_2} = 0, \quad n_1 \geq 0, n_2 \geq 0, \\ S_{n_1 0} &= S_{0 n_2} = 0, \quad n_1 \geq 0, n_2 \geq 0, \\ S_n &= \sum_{1 \leq j \leq n} \xi_j, \quad n \geq 1, \\ X_n(t) &= (\sigma^2 n_1 n_2)^{-1/2} S_{[nt]}, \quad n \geq 1 \text{ and } t \in I^2,\end{aligned}$$

where $[nt] = ([n_1 t_1], [n_2 t_2])$.

THEOREM 3. Let $\{\xi_n, n \in Z^2\}$ be a stationary and ergodic random net with $E\xi_n^2 = 1$. Suppose further that $X_n(t)$ is a strong martingale relative to $\mathcal{F}_t$. Then $\{X_n, n \geq 1\}$ converges weakly, in D^2 , to the two-parameter Wiener process.

REMARKS. (i) There is no loss of generality by considering $\sigma^2 = 1$ in our theorem.

(ii) For the concept of strong martingale see Cairoli-Walsh [4].

(iii) Since X_n is a strong martingale we have $E\xi_n = 0 \forall n$ and $E(\xi_n | \mathcal{F}_{m_1}^{(1)} \vee \mathcal{F}_{m_2}^{(2)}) = 0$ if $n_1 > m_1$ and $n_2 > m_2$.

PROOF. (a) Convergence of finite dimensional distributions. By Theorem 1 it is enough to show that

$$(4.1) \quad (X_n^{(1)}(s^1; I_1), \dots, X_n^{(1)}(s^k; I_k)) \xrightarrow{D} (W^{(1)}(s^1; I_1), \dots, W^{(1)}(s^k; I_k))$$

for arbitrary $s^1, \dots, s^k$ and disjoint intervals $I_1, \dots, I_k$. But (4.1) is implied by Theorem 2 since $(X_n^{(1)}(s^1; I_1), \dots, X_n^{(1)}(s^k; I_k))$ is a R^k -valued martingale.

(b) To prove tightness we can follow Billingsley's arguments [3, pp. 207-208] and observing that

(b1) if $i \neq j$, say $i_1 > j_1$, then

$$E\{\xi_i \xi_j\} = E\{E(\xi_i \xi_j | \mathcal{F}_{j_1}^{(1)} \vee \mathcal{F}_{\min(i_2, j_2)}^{(2)})\} = 0$$

so that $ES_n^2 = n_1 n_2$.

(b2) if $|\xi_0| \leq M < \infty$ with probability 1 then there exists a constant α such that

$$(4.2) \quad ES_n^4 \leq \alpha n_1^2 n_2^2 M^4.$$

Notice the summands of S_n^4 will be of the form $\xi_i \xi_j \xi_k \xi_l$.

Case 1. If $i_1 > j_1 \geq k_1 \geq l_1$

$$E(\xi_i \xi_j \xi_k \xi_l) = E(\xi_j \xi_k \xi_l E(\xi_i | \mathcal{F}_{j_1}^{(1)})) = 0.$$

Case 2. If $i_1 = j_1 > k_1 \geq l_1$

$$E(\xi_i \xi_j \xi_k \xi_l) \begin{cases} = E(\xi_j \xi_k \xi_l E(\xi_i | \mathcal{F}_{k_2}^{(1)} \vee \mathcal{F}_{j_2}^{(2)})) = 0, & i_2 > j_2, \\ \leq M^2 |E(\xi_k \xi_l)| = 0 & \text{if } i_2 = j_2 \text{ and } k \neq l, \\ \leq M^4 & \text{if } i_2 = j_2 \text{ and } k = l, \end{cases}$$

similarly one can estimate all the remaining cases and get the estimate (4.2).

$$(b3) \ E(\max_{\mathbf{k} \leq \mathbf{n}} |S_{\mathbf{k}}|^4) \leq (4/3)^8 E|S_{\mathbf{n}}|^4.$$

This follows from Shorack-Smythe [9, Lemma 1] (cf. equation (4)).

(b4) If $|\xi_0| \leq M < \infty$ then

$$E\left\{\max_{\mathbf{k} \leq \mathbf{n}} |S_{\mathbf{k}}|^4\right\} \leq \alpha' n_1^2 n_2^2 M^4.$$

(b5) Using Billingsley's truncation arguments the proof follows. $\square$

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