CONTRACTIBLY EMBEDDED 2-SPHERES IN $S^2 \times S^2$

YOUN W. LEE

ABSTRACT. We construct contractibly embedded 2-spheres in $S^2 \times S^2$ but not contained in a 4-ball.

1. Introduction. The purpose of this paper is to describe smooth embeddings of S^2 into $S^2 \times S^2$ that are contractible but whose images are not contained in a 4-ball.

These examples show that the following statement is false, which if it were true, may have provided a step in solving the 4-dimensional Poincaré conjecture. Given any two homotopic embeddings of S^2 into $S^2 \times S^2$, there exists an isotopy of one embedding pushing its image into a tubular neighborhood of the other. It is interesting to see what happens to the statement if two homotopic embeddings are replaced with two concordant embeddings.

In the next section, we state some necessary lemmas and construct an example. The proof of the lemmas are given in §3 and more examples are considered in §4.

2. Construction. The construction is based on the following lemmas. We denote M for $S^2 \times S^2$ and K for a smooth embedding of S^2 .

LEMMA 1. *If K is contained in a 4-ball in M , then $H_2(\pi_1(M - K); \mathbb{Z}) = 0$.*

$H_2(\pi_1(M - K); \mathbb{Z})$ is the second homology group of $\pi_1(M - K)$ with coefficients $\mathbb{Z}$. For the definition, see [3, 5 or 10].

LEMMA 2. *Let L be a 2-dimensional knot in S^4 and $S \subset S^4$ a smoothly embedded circle disjoint from L . The L represents a trivial element in $H_2(S^4 - S; \mathbb{Z})$ if and only if S represents an element in the commutator subgroup of $\pi_1(S^4 - L)$.*

Given L and S as in the above lemma, we construct a smoothly embedded 2-sphere, K , in M as follows. Since any two embedded circles in S^4 are isotopic, we may assume that S is the standardly embedded circle in S^4 . The complement of a small open tubular neighborhood of S in S^4 missing L is naturally diffeomorphic to $S^2 \times D_+^2$ in $M = S^2 \times D_+^2 \cup S^2 \times D_-^2$. Since L is contained in $S^2 \times D_+^2$, this gives an embedding K of S^2 into M .

LEMMA 3. *Under the above notations, let $G = \pi_1(S^4 - L)$ and H be the normal closure generated by the element represented by S in G . Then $\pi_1(M - K) \simeq G/H$.*

LEMMA 4. *Under the notations of Lemma 3, if S represents an element in the commutator subgroup of G , then $H_2(\pi_1(M - K); \mathbb{Z}) \simeq H/[G, H]$, where $[G, H]$ is the normal subgroup of G generated by $\{g^{-1}h^{-1}gh = [g, h]: g \in G, h \in H\}$.*

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We now give an embedding of S^2 into M as claimed in the introduction. Let L be the 5-twist-spun trefoil. (See [14 or 9] for the definition.) Then $\pi_1(S^4 - L) \simeq \mathbf{Z} \oplus B$, where B is the binary dodecahedral (icosahedral) group. B is the perfect group of order 120 with the following presentation:

$$B = \langle x, y; (xy)^2 = x^3 = y^5 \rangle.$$

The subgroup H generated by x^3 in B is of order 2 and it is the center of B . We denote G for $\pi_1(S^4 - L)$ and regard H as a subgroup of G .

Let S be a smoothly embedded circle representing x^3 in G . Let K be the embedded 2-sphere in M constructed from L and S as in the paragraph preceding Lemma 3. By Lemma 2, K is contractible in M , since B is perfect.

$$H_2(\pi_1(M - K); \mathbf{Z}) \simeq H/[G, H] \simeq H \simeq \mathbf{Z}_2$$

by Lemmas 3 and 4. Therefore K is not contained in a 4-ball.

3. Proofs of the lemmas. Lemma 1 is obvious because if K is contained in a 4-ball, then $\pi_1(M - K)$ is isomorphic to a 2-dimensional knot group and the second homology group of a knot group (in every dimension) is trivial. (See Chapter 11 of [9].)

PROOF OF LEMMA 2. Let $X = S^4 - L$, then we have the following two exact sequences. The coefficients $\mathbf{Z}$ are omitted.

$$\begin{aligned} \rightarrow H_2(L) \xrightarrow{i} H_2(S^4 - S) \rightarrow H_2(S^4 - S, L) \rightarrow 0, \\ \rightarrow H^1(X) \xrightarrow{j} H^1(S) \rightarrow H^2(X, S) \rightarrow 0. \end{aligned}$$

By the duality (see [12]), $H_2(S^4 - S, L) \simeq H^2(X, S)$. Since $H_2(L) \simeq H^1(X) \simeq H_2(S^4 - S) \simeq H_1(S) \simeq \mathbf{Z}$, $\mathbf{Z}/i(\mathbf{Z}) \simeq \mathbf{Z}/j(\mathbf{Z})$. Therefore i is the trivial homomorphism if and only if j is. But j is trivial if and only if S represents an element in the commutator subgroup of $\pi_1(S^4 - L)$.

Lemma 3 is obvious from the Van Kampen theorem (see [5]).

PROOF OF LEMMA 4. The lemma can be proved topologically but we use an exact sequence derived from a spectral sequence. Associated to a short exact sequence of groups,

$$1 \rightarrow H \rightarrow G \rightarrow G/H \rightarrow 1,$$

there is an exact sequence,

$$H_2(G) \rightarrow H_2(G/H) \rightarrow H/[G, H] \rightarrow H_1(G) \rightarrow H_1(G/H) \rightarrow 0.$$

(See [13 or 2].) The homology is over $\mathbf{Z}$.

From the assumptions of the lemma, $H_1(G) \simeq H_1(G/H) \simeq \mathbf{Z}$ and $H_2(G) = 0$, since G is a knot group. This completes the proof.

4. More examples. Let $T_{p,q}$ be the torus knot of type (p, q) . If a $+1$ (-1) surgery is done on $T_{p,q}$, we get a homology 3-sphere N . (See [4 or 9].) By [7, 8 and 11], $\pi_1(N)$ has a nontrivial center. Let c' be a nontrivial element in the center of $G' = \pi_1(N)$ and $f: G = \pi_1(S^3 - T_{p,q}) \rightarrow G'$ be the projection. Since G' is perfect, we may write $c' = \Pi[a'_i, b'_i]$, where a'_i and $b'_i \in G'$, $k = 1, 2, \dots, n$. Choose elements a_i and b_i , $i = 1, 2, \dots, n$, such that $a_i \in f^{-1}(a'_i)$ and $b_i \in f^{-1}(b'_i)$. Let $c = \Pi[a_i, b_i]$. Denote H for the normal closure generated by c in G and H' for the

normal closure generated by c' in G' . f induces an epimorphism of $H/[G, H]$ onto $H'/[G', H'] \simeq H' \neq 0$.

Now let L be the 2-dimensional spun knot of $T_{p,q}$. $\pi_1(S^3 - T_{p,q}) \simeq \pi_1(S^4 - L)$. (See [1 or 9].) We identify G with $\pi_1(S^4 - L)$. Let S be an embedded circle in $S^4 - L$ representing c . Then L and S produce a contractibly embedded 2-sphere in M not contained in a 4-ball.

ADDENDUM. The following example is pointed out by A. Brunner (cf. K. Murasugi, *On a group that cannot be the group of a 2-knot*, Proc. Amer. Math. Soc. **64** (1977), p. 154). Let L be the spun trefoil knot. $\pi_1(S^4 - L) \simeq \langle a, b | a^2 = b^3 \rangle$. Let S be the embedded circle representing $[a, b]^2$. L and S produce an example.

REFERENCES

1. E. Artin, *Zur Isotopie zweidimensionaler Flächen in R_4* , Hamburg Abh. **4** (1925), 174–177.
2. A. Babakhanian, *Cohomological methods in group theory*, Marcel, New York, 1972.
3. H. Hopf, *Fundamentalgruppe und zweite Bettische Gruppe*, Comment. Math. Helv. **14** (1941–42), 257–309.
4. R. Kirby, *A calculus for framed links in S^3* , Invent. Math. **45** (1978), 35–56.
5. S. Mac Lane, *Homology*, Springer-Verlag, Berlin and New York, 1967.
6. W. Massey, *Algebraic topology, an introduction*, Harcourt, Brace and World, New York, 1967.
7. J. Milnor, *Groups which act on S^n without fixed point*, Amer. J. Math. **79** (1957), 623–630.
8. L. Moser, *Elementary surgery along a torus knot*, Pacific J. Math. **38** (1971), 737–747.
9. D. Rolfsen, *Knots and links*, Publish or Perish, Berkeley, Calif., 1976.
10. J. Rotman, *Notes on homological algebra*, Van Nostrand, New York, 1970.
11. H. Seifert, *Topologie dreidimensionaler gefaster Räume*, Acta Math. **60** (1933), 145–238.
12. E. Spanier, *Algebraic topology*, McGraw-Hill, New York, 1966.
13. J. Stallings, *Homology and central series of group*, J. Algebra **2** (1965), 170–181.
14. E. Zeeman, *Twisting spun knots*, Trans. Amer. Math. Soc. **115** (1965), 471–495.

DEPARTMENT OF MATHEMATICS, MICHIGAN STATE UNIVERSITY, EAST LANSING, MICHIGAN 48824

Current address: Science Division, University of Wisconsin-Parkside, Kenosha, Wisconsin 53141