THE COHOMOLOGY OF BSO_n AND BO_n WITH INTEGER COEFFICIENTS

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ABSTRACT. The cohomology rings of BSO_n and BO_n , with integer coefficients, are given by generators and relations expressed in terms of characteristic classes.

Recently I wished to know the cohomology, with integer coefficients, of BO_n in terms of characteristic classes. I was unable to find this in the literature but found it could be calculated by standard techniques. The results of these calculations and brief proofs are given below. For the sake of completeness and because very little extra effort is required, I have included a variety of well-known results including the existence of Stiefel-Whitney, Chern and Pontrjagin classes—starting from the Thom isomorphism and basic facts about vector bundles, (for example, see [1 and 2]). The only new results in this paper are fairly simple, direct descriptions of $H^*(BO_n; Z)$ and $H^*(BSO_n; Z)$ in terms of Pontrjagin and Stiefel-Whitney classes. These results are implicitly contained in papers of Emery Thomas [3, 4, 5 and 6] and similar results have been obtained by Mark Feshbach [7].

1. Statement of theorems. Let $G_n = U_n, O_n$ or SO_n , let $\zeta_n = \zeta(G_n)$ be the universal n -plane bundle over BG_n and let E_n and S_n be the total space of ζ_n and the space of nonzero vectors of ζ_n , respectively. Recall, one may identify E_n and S_n with BG_n and BG_{n-1} so that the inclusion $S_n \subset E_n$ corresponds to the classifying map $BG_{n-1} \rightarrow BG_n$ of ζ_{n-1} plus the trivial line bundle. We denote by i and j the inclusion maps $i: BG_{n-1} \rightarrow BG_n$ and $j: BG_n \rightarrow (BG_n, BG_{n-1})$. Let $V_n \in H^{\bar{n}}(BG_n, BG_{n-1}; \Lambda)$ be the Thom class where $\bar{n} = n$ or $2n$ according as $G_n = O_n, SO_n$ or $G_n = U_n$ and $\Lambda = Z$ or Z_2 as $G_n = SO_n, U_n$ or $G_n = O_n$. Let $X_n = j^*V_n \in H^{\bar{n}}(BG_n; \Lambda)$.

We denote by ρ and δ the maps in the long exact sequence,

$$(1.1) \quad \rightarrow H^q(X; Z) \xrightarrow{2} H^q(X; Z) \xrightarrow{\rho} H^q(X; Z_2) \xrightarrow{\delta} H^{q+1}(X; Z) \rightarrow$$

which comes from the sequence $0 \rightarrow Z \xrightarrow{2} Z \rightarrow Z_2 \rightarrow 0$.

THEOREM 1.2. *There are unique classes $w_q = w_q(O_n) \in H^q(BO_n; Z_2)$ such that $w_q(O_{n-1}) = i^*w_q(O_n)$, $q < n$, and $w_n(O_n) = X_n(O_n)$. Furthermore,*

$$w_q V_n = \text{Sq}^q V_n, \quad H^*(BO_n; Z_2) = Z_2[w_1, w_2, \dots, w_n]$$

and under the Whitney sum map

$$BO_m \times BO_n \rightarrow BO_{m+n}, \quad w_q \rightarrow \sum w_i \otimes w_{q-i} \quad (w_0 = 1).$$

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THEOREM 1.3. Let $w_q = w_q(\zeta(SO_n)) \in H^q(BSO_n; Z_2)$. Then $w_1 = 0$ and

$$H^*(BSO_n; Z_2) = Z_2[w_2, w_3, \dots, w_n].$$

THEOREM 1.4. There are unique classes $c_q = c_q(U_n) \in H^{2q}(BU_n; Z)$ such that $i^*c_q(U_n) = c_q(U_{n-1})$, $q < n$, and $c_n(U_n) = X_n(U_n)$.

Furthermore

$$H^*(BU_n; Z) = Z[c_1, c_2, \dots, c_n],$$

under the Whitney sum map $BU_m \times BU_n \rightarrow BU_{n+m}$,

$$c_q \rightarrow \sum c_i \otimes c_{q-i}$$

and $\rho c_q(U_n) = w_{2q}(\zeta(U_n))$.

THEOREM 1.5. Let $p_q = (-1)^q c_{2q}(\zeta(SO_n) \otimes C) \in H^{4q}(BSO_n; Z)$ where $\zeta(SO_n) \otimes C$ denotes the complexification of $\zeta(SO_n)$. Then $H^*(BSO_n; Z) = \bar{\mathcal{R}}_n / \mathcal{I}_n$ where

$$\bar{\mathcal{R}}_n = Z[p_1, \dots, p_{[(n-1)/2]}, X_n, \delta(w_{2i_1} w_{2i_2} \cdots w_{2i_l}) | 0 < i_1 < \cdots < i_l \leq [(n-1)/2]]$$

and $\mathcal{I}_n$ is the ideal generated by the relations: $2\delta(w_{2i_1} \cdots w_{2i_l}) = 0$, $X_n = \delta w_{2k}$ if $n = 2k + 1$, and, if for $I = \{i_1, i_2, \dots, i_s\}$, $w(2I) = w_{2i_1} w_{2i_2} \cdots w_{2i_s}$ and $p(I) = p_{i_1} p_{i_2} \cdots p_{i_s}$, then

$$\delta w(2I) \delta w(2J) = \sum_{k \in I} (\delta w_{2k}) p((I - \{k\}) \cap J) \delta(w(2((I - \{k\}) \cup J - (I - \{k\}) \cap J))).$$

Furthermore, $\rho(p_q) = w_{2q}^2$, $\rho\delta = \text{Sq}^1$, and if n is even, $p_{n/2} = X_n^2$. Under Whitney sum, $X_{n+m} \rightarrow X_m \otimes X_n$ (see 1.6 for the behavior of p_q under Whitney sum).

THEOREM 1.6. Let

$$p_q = (-1)^q c_{2q}(\zeta(O_n) \otimes C) \in H^{4q}(BO_n; Z).$$

Then, $H^*(BO_n; Z) = \mathcal{R}_n / \mathcal{I}_n$ where

$$\mathcal{R}_n = Z[p_1, \dots, p_{[n/2]}, \delta(w_1^\epsilon w_{2i_1} \cdots w_{2i_l}) | \epsilon = 0 \text{ or } 1, 0 < i_1 < \cdots < i_l \leq [n/2]],$$

and $\mathcal{I}_n$ is the ideal generated by the relations: $2\delta(w_1^\epsilon w_{2i_1} \cdots w_{2i_l}) = 0$, $\delta(w_1 w_n) = 0$ and $(\delta w_n)^2 = p_{n/2} \delta w_1$, if n is even. Also, if for $I = \{\epsilon/2, i_1, \dots, i_s\}$, $w(2I) = w_1^\epsilon w_{2i_1} \cdots w_{2i_s}$ and $p(I) = (\delta w_1)^\epsilon p_{i_1} \cdots p_{i_s}$, then $\mathcal{I}_n$ contains relations for $\delta w(2I) \delta w(2J)$ given by the formulas in 1.5.

Furthermore $\rho(p_q) = w_{2q}^2$, $\rho\delta = \text{Sq}^1$ and under Whitney sum, $p_q \rightarrow \sum r_{2q-j} \otimes r_j$ where $r_{2s} = p_s$ and $r_{2s+1} = (\delta w_{2s})^2 + p_s \delta w_1$.

2. Proofs. We prove 1.2–1.6 by induction on n . For the inductive step we use the following exact sequence; $H^*(BG_n)$ denotes cohomology with Z_2 coefficients when $G_n = O_n$ and Z coefficients otherwise.

$$(2.1) \quad \begin{array}{ccccccc} \rightarrow H^{q-1}(BG_{n-1}) & \xrightarrow{\delta^*} & H^q(BG_n, BG_{n-1}) & \xrightarrow{j^*} & H^q(BG_n) & \xrightarrow{i^*} & H^q(BG_{n-1}) \rightarrow \\ & & \wr \cup V_n & \nearrow & & & \\ & & H^{q-\bar{n}}(BG_n) & \swarrow & \cup X_n & & \end{array}$$

The map $\cup V_n$ is an isomorphism by the Thom isomorphism theorem.

For $n = 0$, BG_n is a point and 1.2–1.6 are trivial. Suppose 1.2–1.6 are true for $n - 1$.

PROOF OF 1.2. $H^{q-n}(BO_n) = 0$ for $q < n$ and hence $i^*: H^q(BO_n) \approx H^q(BO_{n-1})$ for $q < n$. Hence there are unique classes $w_q \in H^q(BO_n)$ such that $i^*w_q = w_q$, $q < n$, and w_n is defined to be X_n . Since $H^*(BO_{n-1}) = Z_2[w_1, \dots, w_{n-1}]$, i^* is an epimorphism and thus $\cup X_n$ is a monomorphism. Induction q now shows that $H^*(BO_n) = Z_2[w_1, \dots, w_n]$. Under the bundle map, $\zeta_{n-1} \times R \rightarrow \zeta_n$, the Thom class $V_n \in H^n(BO_n, BO_{n-1})$ goes into $V_{n-1} \otimes U \in H^n((BO_{n-1}, BO_{n-2}) \times (R, R - 0))$, where $U \in H^1(R, R - 0)$ is the generator. Thus for $i < n$, $\text{Sq}^i V_n$ goes into $\text{Sq}^i V_{n-1} \otimes U = w_i V_{n-1} \otimes U$. Hence $\text{Sq}^i V_n = w_i \cup V_n$. For $i = n$, $\text{Sq}^n V_n = V_n^2 = (i^* V_n) V_n = w_n V_n$. From commutativity of the diagram

$$\begin{array}{ccc} BO_m \times BO_{n-m-1} & \rightarrow & BO_{n-1} \\ \downarrow & & \downarrow \\ BO_m \times BO_{n-m} & \rightarrow & BO_n \end{array}$$

$w_q \in H^q(BO_n)$, $q < n$, goes into $\sum w_{q-p} \otimes w_p$. Under the Whitney sum map, at the Thom space level, $V_n \rightarrow V_m \otimes V_{n-m}$ and hence $w_n = X_n \rightarrow w_m \otimes w_{n-m}$.

PROOF OF 1.3. For $n = 1$, $w_1 = 0 \in H^1(BSO_1) = H^1(\text{pt})$. The inductive step is the same as the inductive step above that shows $H^*(BO) = Z_2[w_1, \dots, w_n]$.

PROOF OF 1.4. The proof is the same as for 1.2. As part of the induction one proves that $\rho(c_q) = w_{2q}$ from the fact that $\rho V(U_n)$ is the Thom class with Z_2 coefficients of $\zeta(U_n)$ as a real $2n$ bundle.

PROOF OF 1.5. The following is an immediate consequence of the exact sequence 2.1.

LEMMA 2.2. Suppose $H^*(X; Z)$ has the property that if $u \in H^*(X; Z)$ and $ku = 0$ for some k , then $2u = 0$. Then for $v \in H^*(X; Z_2)$, $\delta v = 0$ if and only if $\text{Sq}^1 v = 0$ ($\text{Sq}^1 = \rho\delta$). Furthermore, if $T \subset H^*(X; Z)$ is the torsion subgroup, $T = \delta H^*(X; Z_2)$ and $\rho|_T: T \rightarrow H^*(X; Z_2)$ is an injection.

We next show that $\rho(p_i) = w_{2i}^2$.

$$\begin{aligned} \rho(p_i) &= \rho(c_{2i}(\zeta_n \otimes C)) = w_{4i}(\zeta_n \otimes C) \\ &= w_{4i}(2\zeta_n) = w_{2i}^2(\zeta_n). \end{aligned}$$

In the following, $H^*(BSO_n) = H^*(BSO_n; Z)$. The generators of $\bar{\mathcal{R}}_n$, p_1 , p_2 , etc., denote elements in $H^*(BSO_n)$ and hence define a map $\theta: \bar{\mathcal{R}}_n \rightarrow H^*(BSO_n)$. We show that $\rho\theta(\bar{\mathcal{I}}_n) = 0$.

$$\begin{aligned} (\text{Sq}^1 w_{2k})V_n &= \text{Sq}^1(w_{2k}V_n) + w_{2k}\text{Sq}^1 V_n = \text{Sq}^1 \text{Sq}^{2k} V_n \\ &= \text{Sq}^{2k+1} V_n = w_{2k+1}V_n. \end{aligned}$$

Hence $\text{Sq}^1 w_{2k} = w_{2k+1}$. Thus $\rho X_{2k+1} = w_{2k+1} = \rho\delta w_{2k}$.

The fact that $\rho\theta$ takes the relations for $\delta w(2I)\delta(w(2J))$ to zero follows by induction on the length of I and the following identities:

$$\begin{aligned} \text{Sq}^1 X \text{Sq}^1(YZ) + \text{Sq}^1(XY)\text{Sq}^1 Z + \text{Sq}^1(XZ)\text{Sq}^1 Y &= 0, \\ p_i \text{Sq}^1 X &= \text{Sq}^1(Xw_{2i}^2). \end{aligned}$$

Now suppose n is even. Note $\bar{\mathcal{R}}_n/\bar{\mathcal{I}}_n = (\bar{\mathcal{R}}_{n-1}/\bar{\mathcal{I}}_{n-1})[X_n]$. By inductive hypothesis, $i^*\theta$ and hence $i^*: H^*(BSO_n) \rightarrow H^*(BSO_{n-1})$ is an epimorphism.

Therefore we have an exact sequence:

$$(2.3) \quad 0 \rightarrow H^{q-n}(BSO_n) \xrightarrow{\cup X_n} H^q(BSO_n) \xrightarrow{i^*} H^q(BSO_{n-1}) \rightarrow 0$$

Induction on q shows that θ is an epimorphism. We wish to show that $H^*(BSO_n)$ satisfies 2.2, that is, its torsion is a Z_2 vector space. Since $2\delta = 0$, we must show that the subalgebra generated by $\{p_i, X_n\}$ has no torsion. This follows from 2.3 and induction on q , that is, such a torsion element must be zero in $H^*(BSO_{n-1})$ and hence divisible by X_n . Lemma 2.2 and $\rho\theta\mathcal{I}_n = 0$ yield $\theta\mathcal{I}_n = 0$.

The sequence 2.3 and induction on q now show that $\overline{\mathcal{R}}_n/\mathcal{I}_n = H^*(BSO_n)$.

The relation $X_n^2 = p_{n/2}$ is obtained by tracing $X(SO_{2n})$ around the diagram

$$\begin{array}{ccc} BSO_n & \rightarrow & BU_n \\ \downarrow & & \downarrow \\ BSO_n \times BSO_n & \rightarrow & BSO_{2n} \end{array}$$

(the sign choice, $p_q = (-1)^q c_{2q}$ comes into play).

Now suppose n is odd. As we saw above, $\rho X_n = \rho\delta w_{n-1}$ and hence $X_n = \delta w_{n-1} + 2u$, $u \in H^{n-1}(BSO_n)$.

$$2X_n V_n = 2V_n^2 = 0.$$

Hence $2X_n = 0$ and $4u = 0$. But $H^{n-1}(BSO_n) \approx H^{n-1}(BSO_{n-1})$ and hence $2u = 0$. Therefore $X_n = \delta w_{n-1}$.

Let $\theta: \overline{\mathcal{R}}_n \rightarrow H^*(BSO_n)$ as above. We show that $\text{image}(i^*\theta) = \text{kernel } \delta^* = \text{image } i^*$ and hence, by induction on q in the exact sequence 2.1, that θ is an epimorphism.

Since $2X_n = 0$, $2V_n \in \text{image } \delta^*$. The only elements in $H^{n-1}(BSO_{n-1})$ not in the image of $i^*\theta$ are multiples of X_{n-1} . Hence $\delta^* X_{n-1} = 2V_n$. Any element of $H^*(BSO_{n-1})$ has the form $u = i^*\theta u_1 + (i^*\theta(u_2))X_{n-1}$. $\delta^* u = \theta(u_2)\delta^* X_{n-1} = 2\theta(u_2)V_n$. Therefore $\delta^* u = 0$ if and only if $2i^*\theta(u_2) = 0$. Suppose $\delta^* u = 0$. Then u_2 can be chosen to be a sum of terms each having a factor $\delta(w_{2i_1} \cdots w_{2i_l})$, $2i_l \leq n-3$,

$$\begin{aligned} \delta(w_{2i_1} \cdots w_{2i_l})X_{n-1} &= \delta(w_{2i_1} \cdots w_{2i_l} \rho X_{n-1}) \\ &= \delta(w_{2i_1} \cdots w_{2i_l} w_{n-1}) \\ &\in \text{image } i^*\theta \end{aligned}$$

Hence $u \in \text{image } i^*$. This completes the proof that θ is an epimorphism.

Note the free part of $\overline{\mathcal{R}}_n/\mathcal{I}_n$, the subalgebra generated the p_i , injects into $H^*(BSO_{n-1})$ and the torsion part consists of elements of order two. Hence $H^*(BSO_n)$ satisfies 2.2 and as above $\theta(\mathcal{I}_n) = 0$.

It follows from 2.2 and 2.4 that to show that θ is an isomorphism, it is sufficient to show that $\rho\theta$ induces an isomorphism from the torsion of $\overline{\mathcal{R}}_n/\mathcal{I}_n$ onto $\text{Sq}^1 H^*(BSO_n; Z_2)$. Note:

$$H^*(BSO_n; Z_2) = Z_2[w_{2i}, \text{Sq}^1 w_{2j} | 2i, 2j + 1 \leq n].$$

If $k = (k_1, \dots, k_n)$ let

$$\begin{aligned} w^k &= \prod w_{2i}^{k_{2i}} \prod (\delta w_{2j})^{k_{2j}+1}, \\ z^k &= \prod p_i^{m_i} \prod (\delta w_{2j})^{k_{2j}+1} \delta \left(\prod w_{2i}^{\epsilon_i} \right) \quad \text{where } k_{2i} = 2m_i + \epsilon_i, \end{aligned}$$

$\epsilon_i = 0$ or 1 . Then $\rho\theta(z^k) = \text{Sq}^1 w^k$. Let U be the set of k with k_{2i} odd for some i and, if i_0 is the largest i with k_{2i} odd, then $k_{2j+1} = 0$ for $j > i_0$. One easily checks that z^k , $k \in U$, span the torsion of $\mathcal{R}_n/I_n$ ($\sum \delta w(2(I-i))\delta w_{2i} = 0$). Also $\text{Sq}^1 w^k$, $k \in U$, span $\text{Sq}^1 H^*(BSO_n; Z_2)$. If $k \in U$ and i_0 is as above, let $\bar{k} = k + (0, \dots, -1, 1, 0, \dots, 0)$ where -1 is in the i_0 place. Note that $\bar{k}$ determines k . Then for $k \in U$, $\text{Sq}^1 w^k = w^{\bar{k}}$ modulo the subspace spanned by monomials w^k , $k \notin U$. Hence $\rho\theta(z^k)$, $k \in U$, is a basis for $\text{Sq}^1 H^*(BSO_n; Z_2)$. Therefore θ is an isomorphism.

PROOF OF 1.6. We first prove

LEMMA 2.4. $H^*(BO_n; Z) = Z[p_1, \dots, p_{[n/2]}] \oplus T$ where T is a Z_2 vector space.

PROOF. Suppose n is odd. Let $p: BSO_n \rightarrow BO_n$ be the covering map and let $f: BO_n \rightarrow BO_1$ correspond to w_1 ($BO_1 = K(Z_2, 1)$). Since n is odd, $\zeta_n \otimes f^* \zeta_1$ is orientable. Let $g: BO_n \rightarrow BSO_n$ classify $\zeta_n \otimes f^* \zeta_1$. Since gp is homotopic to the identity, $g_*: \pi_i(BO_n) \approx \pi_i(BSO_n)$, $i > 1$. Also $f_*: \pi_1(BO_n) \approx \pi_1(BO_1)$. Therefore $g \times f: BO_n \rightarrow BSO_n \times BO_1$ is a homotopy equivalence. Therefore

$$p^*: H^*(BO_n; F) \approx H^*(BSO_n; F) \approx F[p_1, \dots, p_{[n/2]}]$$

for any field not of characteristic two.

$$H^*(BO_n; Z) \approx H^*(BSO_n \times BO_1; Z) \approx H^*(BSO_n; H^*(BO_1; Z)).$$

$H^q(BO_1; Z)$ is Z when $q = 0$ and Z_2 or 0 otherwise. Since the torsion of $H^*(BSO_n)$ is a Z_2 vector space, the same is true of $H^*(BO_n; Z)$. Lemma 2.4, for n odd now follows.

Suppose n is even. Let $\bar{V}_{n+1} \in H^{n+1}(BO_{n+1}, BO_n; Z^t)$ be the Thom class where Z^t denotes twisted integer coefficients. The Thom isomorphism yields an exact sequence:

$$(2.5) \quad \begin{array}{ccccc} \rightarrow H^*(BO_{n+1}; Z^t) & \xrightarrow{\cup \bar{X}_{n+1}} & H^*(BO_{n+1}; Z) & \xrightarrow{i^*} & H^*(BO_n; Z) \rightarrow \\ \downarrow \rho & & \downarrow & & \\ H^*(BO_{n+1}; Z_2) & \xrightarrow{\cup w_{n+1}} & H^*(BO_{n+1}; Z_2) & & \end{array}$$

The map ρ is isomorphic to

$$H^*(BSO_{n+1}; H^*(BO_1; Z^t)) \xrightarrow{\rho^*} H^*(BSO_{n+1}; H^*(BO_1; Z_2))$$

and $\rho: H^*(BO_1; Z^t) \rightarrow H^*(BO_1; Z_2)$ is an injection. Hence ρ and $\cup \bar{X}_{n+1}$ in 2.5 are injections and therefore i^* is an epimorphism with kernel a Z_2 vector space. This yields 2.4 for n even.

Let $\theta: \mathcal{R}_n \rightarrow H^*(BO_n)$ be as before. As in the proof of 1.5, $\rho\theta(I_n) = 0$ and hence $\theta(I_n) = 0$. Also the same method used in 1.5, n odd, proves that $\theta: \mathcal{R}_n/I_n \rightarrow H^*(BO_n)$ is an isomorphism.

Finally we determine how the p_i 's behave under Whitney sum. The complexification of the sum of two bundles is the sum of the complexifications. Hence we have a commutative diagram:

$$\begin{array}{ccc} BO_m \times BO_n & \rightarrow & BO_{n+m} \\ \downarrow & & \downarrow \\ BU_m \times BU_n & \rightarrow & BU_{n+m} \end{array}$$

Thus

$$p_i = (-1)^i c_{2i}(\zeta_{n+m} \otimes C) \rightarrow (-1)^i \sum c_{2i-j}(\zeta_m \otimes C) \otimes c_j(\zeta_n \otimes C)$$

and we must show that

$$c_{2s+1}(\zeta_n \otimes C) = (\delta w_{2s})^2 + p_s \delta w_1.$$

Let $r = c_{2s+1}(\zeta_n \otimes C)$. Since $2H^{4s+2}(BO_n) = 0$, $2r = 0$. Thus by 2.2 it is sufficient to obtain the above formula with coefficients reduced mod two.

$$\begin{aligned} \rho r &= w_{4s+2}(\zeta_n \otimes C) = w_{4s+2}(2\zeta_n) = w_{2s+1}(\zeta_n)^2, \\ \rho((\delta w_{2s})^2 + p_s \delta w_1) &= (\text{Sq}^1 w_{2s})^2 + w_{2s}^2 w_1^2 \\ &= (w_{2s+1} + w_1 w_{2s})^2 + w_{2s}^2 w_1^2 = w_{2s+1}^2. \end{aligned}$$

This completes the proof of 1.6.

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