ON THE INERTIA GROUPS OF FIBRE BUNDLES

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ABSTRACT. A subgroup $\tilde{I}(M \times S^i)$ of the inertia group $I(M \times S^i)$ is defined and shown to lie in $I(C)$ for every fibre bundle $M^n \rightarrow C \rightarrow N^i$. For certain M , examples of nontrivial elements in $\tilde{I}(M \times S^i)$ are constructed using the τ -pairing of Milnor-Munkres-Novikov. For compact mapping tori M_g it is shown that $I(M_g) = I(M \times S^1)$ if $\pi_1 M$ is finite and $\text{Wh}(\pi_1 M) = 0$.

1. Introduction. Recall that an exotic n -sphere is a smooth n -manifold homeomorphic to, but not necessarily diffeomorphic to, S^n . Under connected sum the exotic n -spheres form a group, denoted θ_n . Letting $\Gamma^n = \pi_0 \text{Diff}(S^{n-1})$, the map $\Gamma^n \rightarrow \theta_n$ given by $[f] \rightarrow D^n \cup_f D^n = \Sigma(f)$ is an isomorphism for $n \geq 5$ [13]. The inertia group of a smooth n -manifold M is defined by $I(M) = \{\Sigma \in \theta^n: \Sigma \# M \simeq M\}$, and it is a consequence of the disc theorem of Cerf and Palais that $\Sigma(f) \in I(M)$ if and only if there is a diffeomorphism $F: M \setminus \text{int } D^n \rightarrow M \setminus \text{int } D^n$ satisfying $F|_{\partial D^n} = f$. In this way it is seen that $I(M)$ measures the stability of the differentiable structure of M under perturbations of an n -disc.

Many computations of inertia groups have been performed [6, 9, 10, 12, 15, 17 among others]. In this paper a subgroup $\tilde{I}(M \times S^i)$ of $I(M \times S^i)$ is shown to lie in $I(C)$ for every fibre bundle $M^n \rightarrow C \rightarrow N^i$. Examples are constructed for which $\tilde{I}(M \times S^i) \neq 0$, and in particular for $N = S^1$ it is shown that $I(M \times S^1) = I(C)$ for every C if $\pi_1 M$ is finite and satisfies $\text{Wh}(\pi_1 M) = 0$.

2. A special inertia group. Fix a point $*$ in S^i and define $\tilde{I}(M^n \times S^i) = \{\Sigma \in \theta_{n+i}: \text{there is a diffeomorphism } G: (M \times S^i) \# \Sigma \rightarrow M \times S^i \text{ with } G|M \times \{*\} = \text{identity}\}$. Of course, it suffices that $G(M \times \{*\}) = M \times \{*\}$ since then $((G|M \times \{*\}) \times \text{id})^{-1} \circ G$ restricts to the identity on $M \times \{*\}$. Alone, this special inertia group would be of little interest. However, as the following theorem shows it is a useful intermediary for computing inertia groups of fibre bundles.

THEOREM 1. *Let $M^n \rightarrow C \rightarrow N^i$ be a smooth fibre bundle with fibre M^n and base manifold N^i . Then $\tilde{I}(M \times S^i) \subset I(C)$.*

PROOF. Let $G: (M \times S^i) \# \Sigma \rightarrow M \times S^i$ be a diffeomorphism satisfying $G|M \times \{*\} = \text{identity}$, where $*$ in D^i_+ , $\# \Sigma$ is performed in $M \times D^i_-$, and $S^i = D^i_+ \cup D^i_-$. By the tubular neighborhood and isotopy extension theorems, G can be isotoped to G' satisfying $G'(M \times D^i_+) = M \times D^i_+$. Let $G_+ = G'|M \times D^i_+$ and $G_- = G'|(M \times D^i_-) \# \Sigma$, and observe that the composition $\tilde{G} = G_+^{-1} \circ G_-: (M \times D^i_-) \# \Sigma \rightarrow M \times D^i_-$ restricts to the identity on the boundary. Take

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$D^i \subset N$ and observe that since C is trivial over D^i , C admits a decomposition $C = (M \times D^i) \cup (C \setminus (M \times \text{int } D^i))$.

The diffeomorphism defined by the following diagram

$$\begin{array}{ccc} \Sigma \# C = ((M \times D^i) \# \Sigma) & \cup & (C \setminus (M \times \text{int } D^i)) \\ \tilde{G} \downarrow & & \downarrow \text{identity} \\ C = (M \times D^i) & \cup & (C \setminus (M \times \text{int } D^i)) \end{array}$$

shows $\Sigma \in I(C)$. This completes the proof.

Examples of nontrivial elements in $\tilde{I}(M \times S^i)$ can be constructed easily for manifolds of the form $M^n = N^{n-j} \times S^j$, $j \geq i + 2$. Take a generator $\alpha \in \pi_i(SO(i+1))$ and let $s: SO(m) \rightarrow SO(m+1)$ denote the suspension map. Define a self-diffeomorphism h of $N^{n-j} \times S^j \times S^i$ by $h(x, y, z) = (x, s_*\alpha(z) \cdot y, z)$, select $D^j \subset S^j$ satisfying $(s_*\alpha(z))D^j = D^j$ for all $z \in S^i$, and observe that for $D^n = D^{n-j} \times D^j \subset N^{n-j} \times S^j$, $h|D^n \times S^i$ has the form $h((x, y), z) = (x, s_*\alpha(z) \cdot y, z) = (s_*^{n-j+1}\alpha(z) \cdot (x, y), z)$. For some point $z_0 \in S^i$, $s_*\alpha(z_0) = [\text{identity}]$ and $h|M \times \{z_0\} = \text{identity}$. Suppose $\Sigma(f) \in I(M)$ so there is a diffeomorphism $\tilde{f}: M^n \setminus \text{int } D^n \rightarrow M^n \setminus \text{int } D^n$ with $\tilde{f}|_{\partial D^n} = f$. Letting h_0 denote $h|(M \setminus \text{int } D^n) \times S^i$, the commutator $(\tilde{f}^{-1} \times \text{id})h_0^{-1}(\tilde{f} \times \text{id})h_0$ restricted to the boundary represents the element $\tau_{n-1,i}(f, s_*^{n-j+1}(\alpha)) = \beta$ in the image of the Milnor-Munkres-Novikov pairing $\tau_{n-1,i}: \Gamma^n \otimes \pi_i(SO(n)) \rightarrow \Gamma^{n+i}$ (see p. 834 of [10]). Lemma 1 of [7] shows there is a diffeomorphism $F: (M \times S^i) \# \Sigma(\beta) \rightarrow M \times S^i$ and it is easy to see this diffeomorphism satisfies $F|M \times \{z_0\} = \text{identity}$. This proves

COROLLARY. *Let $M^n = N^{n-j} \times S^j$, $j \geq i + 2$. Then for any $\Sigma(f) \in I(M)$, $\Sigma(\beta)$ belongs to $\tilde{I}(M \times S^i)$ and hence to $I(C)$ for each fibre bundle $M^n \rightarrow C \rightarrow N^i$, where $\beta = \tau_{n-1,i}(f, s_*^{n-j+1}(\alpha))$, α a generator of $\pi_i(SO(i+1))$.*

3. Some calculations of $\tilde{I}(M \times S^i)$. To apply Theorem 1 some specific calculations of $\tilde{I}(M \times S^i)$ are needed. Theorems 2 and 3 provide examples.

THEOREM 2. *If M^n is compact, $n \geq 5$, $\pi_1 M$ a torsion group, and $\text{Wh}(\pi_1 M) = 0$, then $I(M \times S^1) = \tilde{I}(M \times S^1)$.*

PROOF. Let $G: (M \times S^1) \# \Sigma \rightarrow M \times S^1$ be a diffeomorphism and observe that since $\pi_1 M$ is a torsion group, $G_\#(\pi_1 M) = \pi_1 M$ and so G induces a diffeomorphism $\hat{G}: (M \times \mathbb{R}) \# \mathbb{Z}\Sigma \rightarrow M \times \mathbb{R}$ of the infinite cyclic coverings. Since M is compact, $\hat{G}(M \times 0)$ lies above some $M \times t \subset M \times \mathbb{R}$ and the regions A between $\hat{G}(M \times 0)$ and $M \times t$ and B between $\hat{G}(M \times 1)$ and $M \times (t+1)$ are h -cobordisms. By the s -cobordism theorem the vanishing of $\text{Wh}(\pi_1 M)$ implies there are trivializations $\tau: A \rightarrow M \times [t, 0]$ and $T\tau T^{-1}: B \rightarrow M \times [t+1, 1]$, where $T: (x, t) \rightarrow (x, t+1)$ is the covering translation corresponding to the projection $M \times \mathbb{R} \rightarrow M \times S^1$. Then $(T\tau T^{-1}|_B|_A) \circ (\hat{G}|_{(M \times [0, 1]) \# \Sigma})$ projects to a diffeomorphism $H: (M \times S^1) \# \Sigma \rightarrow M \times S^1$ satisfying $H(M \times \{*\}) = M \times \{*\}$, hence $\Sigma \in \tilde{I}(M \times S^1)$. This completes the proof.

Denoting by M_g the mapping torus $M \times [0, 1]/\{(m, 0) \sim (g(m), 1)\}$ where $g \in \text{Diff}(M)$, it follows from Theorems 1 and 2 that $I(M \times S^1) \subset I(M_g)$ for all $g \in \text{Diff}(M)$. Examples of manifolds M satisfying the hypotheses of Theorem 2 and having $I(M \times S^1) \neq 0$ can be constructed using Proposition 1 and Theorem 2 of

[12] and also using Theorem 1 of [7]. For another example observe that Theorem 2 implies $I(\mathbb{C}P(3) \times S^1) = \tilde{I}(\mathbb{C}P(3) \times S^1)$, and by II.10 of [3] $I(\mathbb{C}P(3) \times S^1) = \mathbb{Z}_7 \subset \mathbb{Z}_{28} = \theta_7$. Thus for any mapping torus M with fibre $\mathbb{C}P(3)$, $\mathbb{Z}_7 \subset I(M)$. Note that by Corollary 2 of [11] there are nontrivial mapping tori with fibre $\mathbb{C}P(3)$ since $\pi_0 \text{Diff}(\mathbb{C}P(3)) \neq 0$ by II.11 of [3].

An argument from [11] can be combined with Theorem 2 to yield

COROLLARY. *If M^n is compact, $n \geq 5$, $\pi_1 M$ a torsion group, and $\text{Wh}(\pi_1 M) = 0$, then $I(M \times S^1) = I(M_g)$ for $g \in \text{Diff}(M)$.*

PROOF. Theorems 1 and 2 imply $I(M \times S^1) \subset I(M_g)$. For the reverse inclusion, suppose $G: \Sigma \# M_g \rightarrow M_g$ is a diffeomorphism. Let $f \in \text{Diff}(D^n, \text{rel } \partial D^n)$ represent Σ and define a diffeomorphism $\psi(f) \in \text{Diff}(M)$ by $\psi(f)|D^n = f$ and $\psi(f)|M \setminus \text{int } D^n = \text{identity}$, for some $D^n \subset M$. By Lemma 1 of [2] there is a diffeomorphism $H: M_{g\psi(f)} \rightarrow M_g \# \Sigma$, and by Corollary 4 of [11] the diffeomorphism $GH: M_{g\psi(f)} \rightarrow M_g$ is pseudo-isotopic to $F: M_{g\psi(f)} \rightarrow M_g$ with $F|M = \text{identity}$. Then $FH^{-1}: M_g \# \Sigma \rightarrow M_g$ satisfies $FH^{-1}|M = \text{identity}$, so splitting $M_g \# \Sigma$ and M_g along M and regluing by the identity gives a diffeomorphism $(M \times S^1) \# \Sigma \rightarrow M \times S^1$. This completes the proof.

In order to apply Theorem 2 and its corollary, some information is needed about which finite groups π have $\text{Wh}(\pi) = 0$. If π is finite abelian, $\text{Wh}(\pi) = SK_1(\mathbb{Z}\pi) \oplus U(\mathbb{Z}\pi)$ (p. 202 of [16]). For finite cyclic groups $\mathbb{Z}_q$ it follows from a Theorem of Bass (XI.7.3 of [1]) that $SK_1(\mathbb{Z}\mathbb{Z}_q) = 0$, and work of Bass, Higman, Milnor, and Serre (11.5 of [4]) shows $\text{Wh}(\mathbb{Z}_q) = 0$ if and only if $q = 1, 2, 3, 4$, or 6 . Another Theorem of Bass (6.4 of [14]) shows that $U(\mathbb{Z}\pi) = 0$ if π is finite abelian of exponent 1, 2, 3, 4, or 6. Combining this with calculations of Stein (pp. 206 and 204 of [16]) shows $\text{Wh}(\mathbb{Z}_3^2) = 0$ and $\text{Wh}(\mathbb{Z}_2^k) = 0$ for all k . For nonabelian examples, Milnor applies Bass' theorem (pp. 374 of [14]) to show for the symmetric group S_n that $\text{Wh}(S_n)$ is finite, and Magurn (7.3 of [16]) shows $SK_1(S_n) = 0$ for $n = 4, 5$, or 6 , so $\text{Wh}(S_n) = 0$ for $n = 4, 5$, or 6 , by a result of Wall (pp. 209–210 of [16]). Finally, Magurn has shown (p. 210 of [16]) that $SK_1(\mathbb{Z}D_k) = 0$ for any finite dihedral group D_k , and an easy calculation using 39.5 and 47.15 of [5] shows $\text{Wh}(D_k) = 0$ for $k = 3, 4$, and 6 .

For another calculation of $\tilde{I}(M \times S^i)$, recall Brumfiel's definition [3, Appendix II] of the special inertia group $I_h(M) = \{\Sigma \in \theta_n: \text{there is a diffeomorphism } M \rightarrow M \# \Sigma \text{ homotopic to } \iota\}$, where the PL homeomorphism

$$\iota: M = (M \setminus \text{int } D^n) \cup_{\text{id}} D^n \rightarrow (M \setminus \text{int } D^n) \cup_f D^n = M \# \Sigma$$

is defined by $\iota|M \setminus \text{int } D^n = \text{identity}$ and $\iota|D^n = c(f) = \text{cone map on } f$.

THEOREM 3. *If $n+2 < i$, then $I_h(M \times S^i) \subset \tilde{I}(M \times S^i)$ and so $I_h(M \times S^i) \subset I(C)$ for any fibre bundle $M^n \rightarrow C \rightarrow S^i$.*

PROOF. By hypothesis there is a diffeomorphism $G: (M \times S^i) \# \Sigma \rightarrow M \times S^i$ and a homotopy F_t between $F_0 = \iota$ and $F_1 = G$. Since $n+2 < i$, $F_t|(M \times \{*\} \times I)$ can be smoothed and self-general positioned to an imbedding giving rise to a pseudo-isotopy $\tilde{F}: M \times \{*\} \times I \rightarrow M \times S^i \times I$. By Theorem 2.1 of [8] and the isotopy extension theorem, there is an isotopy $H: M \times S^i \times I \rightarrow M \times S^i \times I$ with $H_0 = \text{identity}$ and $H_1|M = \tilde{F}_1$. Define $\tilde{H}: M \times S^i \rightarrow M \times S^i$ by $\tilde{H}(m, s) = H_1^{-1}(m, s)$.

Then $\bar{H}G: (M \times S^i) \# \Sigma \rightarrow M \times S^i$ satisfies $\bar{H}G|M \times \{*\} = \text{inclusion}$. This completes the proof.

As a final example, consider a diffeomorphism $f: \Sigma \# (M^n \times S^i) \rightarrow M^n \times S^i$. If M is $(n - i + 1)$ -connected and $2i - 3 \geq n$, then f is isotopic to f' with $f'|M \times \{*\} = \text{inclusion}$ for some $*$ in S^i if and only if a single obstruction

$$\alpha(f(M \times \{*\}), M \times \{*\}: M \times S^i) \in \pi_{n-i}$$

vanishes [19]. In particular, if M^n is $(n - i + 1)$ -connected and $2i - 3 \geq n$, then $I(M \times S^i) = \tilde{I}(M \times S^i)$ if $n - i = 4, 5$, or 12 (since then π_{n-i} vanishes [18]).

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