CRITERIA FOR REGIONALLY RECURRENT FLOWS

SABER ELAYDI

ABSTRACT. Regionally recurrent and P -regionally recurrent flows are characterized in purely prolongational notions. An example is given to show that the condition $x \in J(x)$ is not, as many authors asserted, equivalent to regional recurrence.

Introduction. Let (X, T) be a flow, where X is a topological space and T is a topological group. Then a point $x \in X$ is said to be T -regionally recurrent or simply regionally recurrent if for each neighbourhood U of x there exists an extensive subset A of T such that $U \cap Ua \neq \emptyset$ for all $a \in A$ [6]. In [3] the author showed that for a flow (X, T) , a point $x \in X$ is regionally recurrent if and only if $x \in J^P(x)$ for all replete semigroups P in T . An example is provided (Example 14) to show that the condition $x \in J(x)$ is not equivalent to regional recurrence as it was stated in [7] and [8]. However, in the setting of continuous flows, where the phase group is the additive group of real numbers R , a necessary and a sufficient condition for a point $x \in X$ to be regionally recurrent is either one of the following (1) $x \in J^+(x)$ or, (2) $x \in J^-(x)$ or (3) $x \in J(x)$. This is due to the simple structure of replete semigroups in R . Every replete semigroup in R contains either a ray in R^+ or a ray in R^- . Thus if $x \in J^+(x)$, then $x \in J^P(x)$ for all replete semigroups P in R^+ . Furthermore, $x \in J^+(x)$ implies that $x \in J^-(x)$. Consequently, $x \in J^Q(x)$ for all replete semigroups Q in R^- and hence x is regionally recurrent. Similar analysis is available for the cases $x \in J^-(x)$ and $x \in J(x)$. The situation is far more complicated in general flows as is shown in Example 14. In this example we have $x \in J(x)$ for each $x \in X$ but $x \notin J^Q(x)$ for some replete semigroups Q in the phase group T .

In [3] it was shown that $x \in X$ is P -regionally recurrent iff it is P -nonwandering. The same conclusion holds for T -regionally recurrent and T -nonwandering. Hence, the work here is a natural generalization of the work in [1] on nonwandering continuous flows. However, our techniques are different and much simpler.

In addition to the above-mentioned, one purpose of this note is to introduce prolongational techniques which are widely used in Dynamical Systems Theory but surprisingly unknown in the theory of Topological Dynamics. As far as I know it was Hajek [7] who suggested the use of prolongational techniques in Topological Dynamics. R. Knight [8, 9] and Elaydi [2-5], with Kaul followed suit. But the power of these techniques has yet to be shown. Our main references for notations and terminology are [7] and [3].

Received by the editors April 6, 1981 and, in revised form, September 20, 1981.

1980 *Mathematics Subject Classification.* Primary 54H20.

Key words and phrases. Regional recurrence, replete semigroups, prolongation, extensive set, nonwandering.

© 1982 American Mathematical Society
 0002-9939/81/0000-0399/\$02.25

From now on (X, T) denotes a flow, where X is a Hausdorff topological space and T is an abelian topological group. $\mathcal{P}$ and P will denote the set of all replete semigroups in T and a certain replete semigroup in T , respectively.

DEFINITION 1. A subset A of T is P -extensive if $A = S \cap P$ for some extensive set S of T .

LEMMA 2. The following statements are pairwise equivalent for a subset A of P .

- (1) A is P -extensive.
- (2) $A \cap Q \neq \emptyset$ for all $Q \in \mathcal{P}$ and $Q \subset P$.
- (3) $A \cap pQ \neq \emptyset$ for all $Q \in \mathcal{P}$, $Q \subset P$ and $p \in P$.

PROOF. (1) implies (2). Since A is P -extensive, $A = S \cap P$ for some extensive set S in T . Hence $S \cap Q \neq \emptyset$ [6, 6.15]. This implies that $A \cap Q = S \cap P \cap Q = S \cap Q \neq \emptyset$.

(2) implies (3). Let $p \in P$. Let K be a compact subset of T and let $H = p^{-1}K$. Then there exists $t \in T$ such that $Ht = tp^{-1}K \subset Q$. It follows that $tK \subset pQ$. Hence pQ is a replete semigroup in T which is contained in P . Hence $A \cap pQ \neq \emptyset$.

(3) implies (1). Define $S = A \cap (T - P)$. Then clearly $A = S \cap P$. Furthermore, S is an extensive set in T .

DEFINITION 3. A point $x \in X$ is said to be P -regionally recurrent if for each neighbourhood U of x there exists a P -extensive set A such that $U \cap Ua \neq \emptyset$ for all $a \in A$.

A point $x \in X$ is T -regionally recurrent or simply regionally recurrent if it is P -regionally recurrent for all $P \in \mathcal{P}$. A transformation group (X, T) is said to have one of the above properties if each point in X has the property.

THEOREM 4. For any flow (X, T) the following statements are pairwise equivalent.

- (1) The flow is P -regionally recurrent.
- (2) $x \in J^Q(x)$ for all $Q \in \mathcal{P}$, $Q \subset P$ and $x \in X$.
- (3) $J^Q(x) = D^Q(x)$ for all $Q \in \mathcal{P}$, $Q \subset P$ and $x \in X$.
- (4) $D^Q(x) = D^{Q^{-1}}(x)$ for all $Q \in \mathcal{P}$, $Q \subset P$ and $x \in X$.

PROOF. (1) implies (2). Let $Q \in \mathcal{P}$ such that $Q \subset P$ and let U be a neighbourhood of $x \in X$. Then $U \cap Ua \neq \emptyset$ for all $a \in A$, for some P -extensive set A . Then by Lemma 2 for each $q \in Q$ there exists $a \in A \cap qQ$. Hence $x \in \overline{UqQ}$ for each $q \in Q$. This implies that $x \in J^Q(x)$.

(2) implies (3). Since $J^Q(x)$ is invariant, $xQ \subset J^Q(x)$. Hence $D^Q(x) = xQ \cup J^Q(x) = J^Q(x)$.

(3) implies (4). Let $y \in D^{Q^{-1}}(x)$. Then $x \in D^Q(y) = J^Q(y)$. This implies that $y \in J^{Q^{-1}}(x)$. Thus $J^{Q^{-1}}(x) = D^{Q^{-1}}(x)$. Let $z \in D^Q(x) = J^Q(x)$. Then $zQ^{-1} \subset J^Q(x)$. If $z \notin J^{Q^{-1}}(x)$, then $zQ^{-1} \cap J^{Q^{-1}}(x) = zQ^{-1} \cap D^{Q^{-1}}(x) = \emptyset$ which is clearly absurd. Hence $D^Q(x) = D^{Q^{-1}}(x)$.

(4) implies (1). Let $x \in X$ and $Q \in \mathcal{P}$ with $Q \subset P$. Since $xQ \subset D^Q(x) = D^{Q^{-1}}(x)$, for each $q \in Q$ we have $xq^{-1} \in D^Q(x) = D^{Q^{-1}}(x)$. This implies that $x \in D^{Q^{-1}}(x)q = D^{Q^{-1}}(xq) = D^Q(xq)$ for all $q \in Q$. Thus for each neighbourhood U of x and $q \in Q$ we have $x \in \overline{UqQ}$. For each $q \in Q$ choose $b \in qQ$ such that $U \cap Ub \neq \emptyset$. Let A be the set of all such elements b with the above-mentioned properties ranging over all replete semigroups Q contained in P . Then according to

Lemma 2 the set A is P -extensive. Consequently x is P -regionally recurrent. This completes the proof of the theorem.

COROLLARY 5. For any flow (X, T) the following statements are pairwise equivalent.

- (1) The flow (X, T) is regionally recurrent.
- (2) $x \in J^P(x)$ for each $x \in X$ and $P \in \mathcal{P}$.
- (3) $J^P(x) = D^P(x)$ for each $x \in X$ and $P \in \mathcal{P}$.
- (4) $D^P(x) = D^{P^{-1}}(x)$ for each $x \in X$ and $P \in \mathcal{P}$.

LEMMA 6. Let $\{x_i\}$ and $\{y_i\}$ be nets in X such that $x_i \rightarrow x$ and $y_i \rightarrow y$. If for each i , $y_i \in J^P(x_i)$ then $y \in J^P(x)$.

PROOF. The proof follows immediately from [2, 2.7].

THEOREM 7. A flow (X, T) is p -regionally recurrent iff the set of p -regionally recurrent points in X is dense in X .

PROOF. This follows from Lemma 6 and Theorem 4.

COROLLARY 8. A flow (X, T) is regionally recurrent iff the set of regionally recurrent points in X is dense in X .

LEMMA 1.9. If $y \in L^P(x)$ for $P \in \mathcal{P}$, then $J^P(x) \subset J^P(y) = D^P(y)$.

PROOF. Let $z \in J^P(x)$. Since $J^P(xp) = J^P(x)$ for all $p \in \mathcal{P}$, $z \in J^P(xp)$ for each $p \in \mathcal{P}$. There exists a net $xp_i \rightarrow y$, $p_i \in \mathcal{P}$. It follows from Lemma 6 that $z \in J^P(y)$. Hence $J^P(x) \subset J^P(y)$. Furthermore, $y \in L^P(x) \subset J^P(x) \subset J^P(y)$ implies that $J^P(y) = D^P(y)$, since $D^P(y) = yP \cup J^P(y) = J^P(y)$, as $J^P(y)$ is invariant.

THEOREM 10. If $y \in L^Q(x)$ for all $Q \in \mathcal{P}$ contained in P , then y is p -regionally recurrent.

PROOF. Use Lemma 9 and Theorem 4.

COROLLARY 11. If $y \in L^P(x)$ for all $P \in \mathcal{P}$, then y is regionally recurrent.

THEOREM 12. Let (X, T) be a P -regionally recurrent flow. Then the set of P -recurrent points in X is dense in X , provided that T is generative [6].

PROOF. We need only to replace T by Q , where Q is any replete semigroup contained in P , in the proof of [8, 1].

REMARK 1.13. Although the proof of [8, 1] uses wrong equivalences to regionally recurrence and recurrence it can be corrected by a slight modification. We will give an example of a flow in which $x \in L(x)$ for each $x \in X$ and yet it is neither regionally recurrent nor recurrent.

EXAMPLE 14. Let (X, T) be a flow, where $X = R$; the set of real numbers with the usual topology and $T = R \times R$, with the operation $(a, b) + (c, d) = (a + c, b + d)$, where R is the additive group of real numbers. Define the action of T on X by $x(a, b) = x + a + b$ for all $x \in X$ and $(a, b) \in R \times R$. Let $P = \{(x, y) \in R \times R | x \geq 0, y \leq 0\}$. Then for each $x \in X$, $x \in L^P(x) \cap L^{P^{-1}}(x)$. Let $Q = \{(x, y) \in R \times R | x \geq 0, y \geq 0\}$. Then for each $x \in X$ we have $x \notin L^Q(x) =$

$J^Q(x)$. Hence the flow is not regionally recurrent. Furthermore, $x \notin L^S(x)$ for each $x \in X$, where $S = \{(x, y) \in R \times R | x \geq 0, -x/3 \leq y \leq 0\}$ is a replete semigroup contained in P . Thus the flow is not even P -regionally recurrent.

REMARK 15. In the above example the flow is of P -characteristic 0 [2]. Also the example [5, 5.1] is of P -characteristic 0, where $P = R^+$, but not P -regionally recurrent. In the sequel we will discuss a few simple relations between these two notions.

THEOREM 16. *Let (X, T) be a flow of P -characteristic 0. If $L^{Q^{-1}}(x) \neq \emptyset$ for all $x \in X$ and $Q \in \mathcal{P}$ contained in P , then (X, T) is P -regionally recurrent.*

PROOF. Use [2, 3.2] and Theorem 4.

COROLLARY 17. *Let (X, T) be a flow of P -characteristic 0 where $L^{P^{-1}}(x) \neq \emptyset$ for all $x \in X$ and $P \in \mathcal{P}$. Then (X, T) is regionally recurrent.*

THEOREM 18. *Let (X, T) be a flow, where X is assumed to be locally compact and T is generative. Then (X, T) is regionally recurrent and of P -characteristic 0 for all $P \in \mathcal{P}$ iff it is pointwise almost periodic and of Q -characteristic 0 for some $Q \in \mathcal{P}$.*

PROOF. Assume that (X, T) is regionally recurrent and of P -characteristic 0 for all $P \in \mathcal{P}$. Then it follows from [2, 3.4] and Corollary 1.5 that $x \in L^P(x)$ for each $x \in X$ and $P \in \mathcal{P}$. Hence the flow (X, T) is recurrent [2, 3.7]. Now use [6, 7.05] and [2, 3.5] to conclude the proof of the first part. To prove the converse assume that the flow is of Q -characteristic 0 for some $Q \in \mathcal{P}$ and pointwise almost periodic. Then $\overline{xT}$ is compact and minimal for all $x \in X$ [6, 4.10]. We need to show that the flow is of P -characteristic 0 for each $P \in \mathcal{P}$. Let $y \in D^P(x)$, $x, y \in X$ and $P \in \mathcal{P}$. Then there are nets $\{x_i\}$ in X and $\{p_i\}$ in P such that $x_i \rightarrow x$ and $x_i p_i \rightarrow y$. For each i , $x_i p_i \in \overline{x_i P} \subset L^{Q^{-1}}(x) \subset D^{Q^{-1}}(x)$. It follows from [2, 2.7] that $y \in D^{Q^{-1}}(x) = D^Q(x) = \overline{xQ} = \overline{xT} = \overline{xP}$. This completes the proof of the theorem.

REFERENCES

1. S. Ahmad and J. Sarabia, *On non wandering continuous flows*, Funkcial. Ekvac. **21** (1978), 193–201.
2. S. Elaydi, *P-recursion and transformation groups of characteristic 0*, J. Univ. Kuwait (Science) **9** (1982) (to appear).
3. —, *Weakly equicontinuous flows*, Funkcial. Ekvac. **24** (1981).
4. —, *On equicontinuous transformation groups*, Bull. Austral. Math. Soc. **24** (1981), 367–372.
5. S. Elaydi and S. K. Kaul, *On characteristic 0 and locally weakly almost periodic flows*, Preprint 42, Univ. of Regina, Canada, 1980.
6. W. H. Gottschalk and G. A. Hedlund, *Topological dynamics*, Amer. Math. Soc. Colloq. Publ., vol. 36, Amer. Math. Soc., Providence, R. I., 1955.
7. O. Hajek, *Prolongations in topological dynamics*, Seminar on Differential Equations and Dynamical Systems II, Lecture Notes in Math., vol. 44, Springer-Verlag, Berlin and New York, 1970, pp. 79–89.
8. R. Knight, *The center of a transformation group*, Proc. Conf. Nonlinear Phenomena in Math. (Univ. of Texas, Arlington, June 1980) (to appear).
9. —, *Prolongationally stable transformation groups*, Math. Z. **161** (1978), 189–194.

DEPARTMENT OF MATHEMATICS, KUWAIT UNIVERSITY, P.O. BOX 5969, KUWAIT