SHORTER NOTES

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A CLOSED INFINITE DIMENSIONAL REFLEXIVE LEFT INVARIANT SUBSPACE OF $L_\infty(G)$

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ABSTRACT. We give an example of a locally compact group G , for which $L_\infty(G)$ has a closed infinite dimensional reflexive left invariant subspace.

In a recent paper I. Glicksberg [2] showed that if G is a compact or abelian group then every closed left invariant reflexive subspace of $L_\infty(G)$ must be finite dimensional. The purpose of this note is to exhibit a nonabelian, noncompact group for which $L_\infty(G)$ has an infinite dimensional closed left translation invariant reflexive subspace.

Let H be an infinite dimensional Hilbert space and $B(H)$ the bounded operators on H . Let G be the group of unitary operators in $B(H)$ with the discrete topology. Then the map $\pi: G \rightarrow B(H)$ is a continuous unitary representation of G on H . We can represent $l_1(G)$ on H in the usual way by defining

$$\pi(f)\xi = \sum_{x \in G} f(x)\pi(x)\xi \quad \text{for all } \xi \in H.$$

Now fix $\xi \in H$, $\xi \neq 0$. We claim that $\pi(l_1(G))\xi = H$. For let $\eta \in H$, $\eta \neq 0$. Then there exists $x_0 \in G$ such that $\pi(x_0)\xi/\|\xi\| = \eta/\|\eta\|$. Define f on G by $f(x_0) = \|\eta\|/\|\xi\|$ and $f(x) = 0$ if $x \neq x_0$. Then $f \in l_1(G)$ and

$$\pi(f)\xi = \sum_{x \in G} f(x)\pi(x)\xi = f(x_0)\pi(x_0)\xi = \eta.$$

Since π is a continuous representation of $l_1(G)$ on H , the map $f \rightarrow \pi(f)\xi$ is a continuous map of $l_1(G)$ onto H . Let $M = \{f \in l_1(G): \pi(f)\xi = 0\}$. Then M is closed. Also since $\pi(xf) = \pi(x^{-1})\pi(f)$, M is closed under left translation.

Let $\Phi: l_1(G)/M \rightarrow H$ be the quotient map. Then Φ is a one-to-one continuous surjection and so by the open mapping theorem it is bicontinuous. It follows that

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the adjoint map $\Phi^*: H^* \rightarrow (l_1(G)/M)^*$ is one-to-one and bicontinuous. By [1, II4.28b] $(l_1(G)/M)^*$ is isometrically isomorphic with

$$M^\perp = \left\{ h \in l_\infty(G) : \sum_{x \in G} f(x)h(x) = 0 \text{ for all } f \in M \right\}.$$

We show that $M^\perp$ is a closed left invariant infinite dimensional reflexive subspace of $l_\infty(G)$. That $M^\perp$ is closed follows from the fact that it is the annihilator of M . The left invariance of $M^\perp$ is a consequence of the left invariance of M . For if $h \in M^\perp$, $x_0 \in G$ and $f \in M$ then

$$\sum_{x \in G} x_0 h(x) f(x) = \sum_{x \in G} h(x_0 x) f(x) = \sum_{x \in G} h(x) f(x_0^{-1} x) = \sum_{x \in G} h(x)_{x_0^{-1}} f(x) = 0.$$

Finally, since $M^\perp$ is bicontinuously isomorphic with H^* , we have that $M^\perp$ is infinite dimensional and reflexive.

REFERENCES

1. N. Dunford and J. T. Schwartz, *Linear operators*, Part I, Interscience, New York, 1958.
2. I. Glicksberg, *Reflexive invariant subspaces of $L_\infty(G)$ are finite dimensional*, Math. Scand. **47** (1980), 308–310.

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