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MINIMIZING SETUPS FOR CYCLE-FREE ORDERED SETS

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ABSTRACT. A machine performs a set of jobs one at a time subject to a set of precedence constraints. We consider the problem of scheduling the jobs to minimize the number of "setups".

Suppose a single machine is to perform a set of jobs, one at a time; a set of precedence constraints prohibits the start of certain jobs until some other jobs are already completed. Any job which is performed immediately after a job which is not constrained to precede it, however, requires a "setup"—entailing some fixed additional cost. The problem is *schedule the jobs to minimize the number of setups*.

It is common to render "a set of precedence constraints on a set of jobs" as "an antisymmetric and transitive binary relation on a set," that is, "a (partial) ordering on a set." In this analogy a "schedule satisfying the precedence constraints" becomes "a linear extension of the ordered set" (of all jobs). The problem of minimizing the number of setups can be entirely recast as a problem concerning linear extensions of an ordered set. The problem itself is attributed in [**2**] to Kuntzmann (cf. [**6**]). Progress on the problem can be found in several papers including [**3**, **4**, and **7**] and recently W. R. Pulleyblank [**7**] has shown that this problem belongs to that class of problems whose complexity is described as NP-hard.

For elements a, b of an ordered set $(P, \leqslant)$—simply written as P—we say that b *covers* a if $a < b$ in P and $a \leqslant c < b$ implies $a = c$. Let L be a linear extension of P; that is, a total ordering of the underlying set of P such that $a < b$ in L whenever $a < b$ in P. A 'setup for L' is an ordered pair (a, b) of elements of P for which b covers a in L but $a \nless b$ (and hence also $a \ngtr b$) in P. Let $s_L(P)$ count the number of such ordered pairs and let

$$s(P) = \min\{s_L(P) \mid L \text{ is a linear extension of } P\}.$$

The problem is *construct a linear extension L of the ordered set P for which* $s_L(P) = s(P)$.

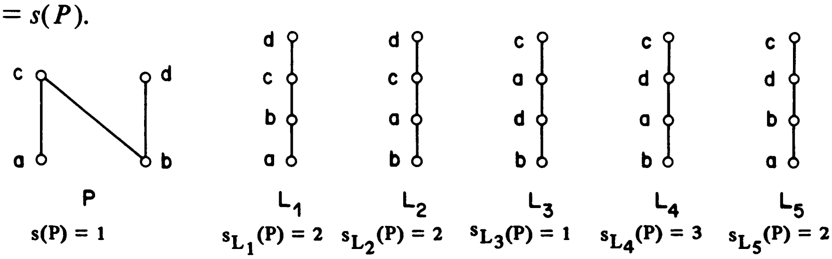

FIGURE 1

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Any linear extension L of P can be obtained by partitioning P into chains (linearly ordered subsets) $C_1, C_2, \ldots, C_m$ such that $x < y$ in L if either $x < y$ in P, or $x \in C_i$ and $y \in C_j$, where $i < j$. In particular, L is the *linear sum* of chains

$$L = C_1 \oplus C_2 \oplus \cdots \oplus C_m.$$

If the greatest element $\max(C_i)$ of C_i is not below the least element $\min(C_{i+1})$ of C_{i+1} in P, then $(\max(C_i), \min(C_{i+1}))$ is a setup for L. Evidently, $s_L(P) \leqslant m - 1$ and if $\max(C_i) \nleqslant \min(C_{i+1})$ for each $i = 1, 2, \ldots, m - 1$, then $s_L(P) = m - 1$. According to Dilworth's theorem [5], the smallest number of chains into which P can be partitioned is equal to the *width* $w(P)$ of P—the size of a maximum-sized antichain. Therefore, $s(P) \geqslant w(P) - 1$.

Of course, equality does not in general obtain. Indeed, a partition $C_1, C_2, \ldots, C_{w(P)}$ of P into chains can be arranged to form a linear extension of P only if there is a permutation ρ of $\{1, 2, \ldots, w(P)\}$ such that $\rho(i) < \rho(j)$ implies $x \ngtr y$ for any $x \in C_{\rho(i)}$ and $y \in C_{\rho(j)}$. No such permutation could exist if there were a subset (say, $\{C_1, C_2, \ldots, C_n\}$) of the partition, and elements $x_i, y_i \in C_i$, $i = 1, 2, \ldots, n$, satisfying

$$y_1 < x_1, x_1 > y_2, y_2 < x_2, x_2 > y_3, \ldots, x_{n-1} > y_n, y_n < x_n, x_n > y_1.$$

An ordered set $\{x_1, y_1, x_2, y_2, \ldots, x_n, y_n\}$ of size $2n$, $n \geqslant 2$, with these comparabilities, and no others, is called an *alternating* $2n$-*cycle*, or more briefly a $2n$-*cycle* (see Figure 2).

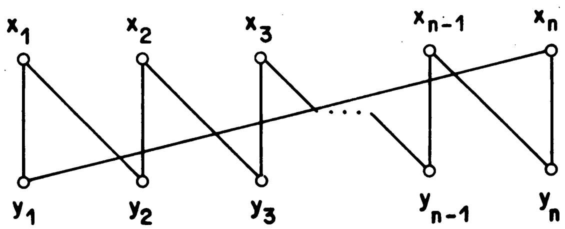


FIGURE 2

The ordered sets shown in Figure 3 are cycle-free, that is, contain no subset isomorphic to an alternating $2n$-cycle.

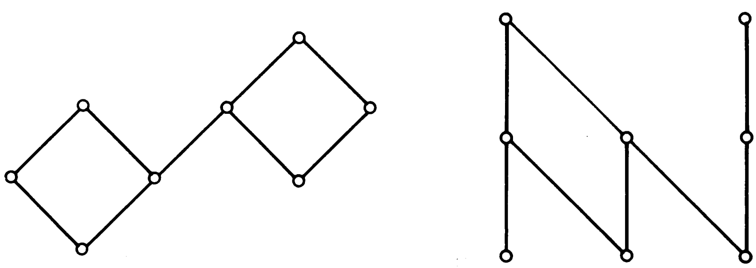


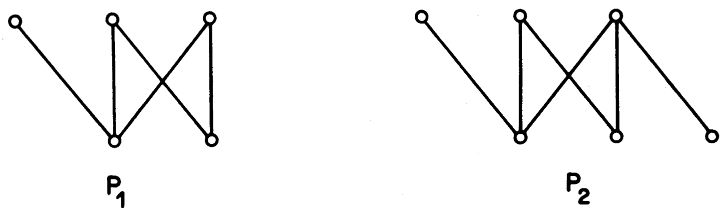
FIGURE 3

The principal result of this paper is

THEOREM. *Let P be an ordered set without alternating cycles. Then $s(P) = w(P) - 1$.*

The case where P has length two (that is, P has no three-element chain) is particularly easy to verify. We proceed by induction on the size of P: if P contains an isolated element a then $w(P - \{a\}) = w(P) - 1$ and clearly $s(P) = s(P - \{a\}) + 1$. Otherwise, as P is cycle-free there is an element b comparable with precisely one other element, say, $b < c$. Again if $w(P - \{b\}) = w(P) - 1$ then the induction hypothesis applies; otherwise, $w(P - \{b\}) = w(P)$ and, indeed, $w(P - \{b, c\}) = w(P) - 1$. Finally, $s(P) = s(P - \{b, c\}) + 1$, so in any case, $s(P) = w(P) - 1$.

Before we turn to the proof of the theorem, note from the ordered sets illustrated in Figure 4 that the converse of the theorem cannot hold.



$s(P_i) = 2 = w(P_i) - 1 \quad (i = 1, 2)$

FIGURE 4

Proof of the theorem. We proceed by induction on $m = w(P)$. Let $C_1, C_2, \ldots, C_m$ be a sequence of *maximal* chains of P such that

$$P = \bigcup_{i=1}^{m} C_i.$$

(Such a sequence can always be obtained by extending each of the m chains in a partition of P by width-many chains.)

Let $x, y, z \in C_i$ with $x < y < z$ and suppose that for some j, $\{x, y, z\} \cap C_j = \{y\}$. Then some element x' in C_j must be noncomparable to x, else the addition of x would extend C_j; similarly there must be an element z' of C_j noncomparable to z. But then $\{x, z, x', z'\}$ is a 4-cycle, contradicting the hypothesis of the theorem. It follows that, for any i and j and any $y \in C_i \cap C_j$, either $\{x \in C_i \cup C_j \mid x \leqslant y\}$ is a chain or $\{z \in C_i \cup C_j \mid z \geqslant y\}$ is a chain.

For each i, let

$$P_i = C_i - \bigcup_{j \neq i} C_j.$$

Then $P_i \neq \varnothing$ for each $i = 1, 2, \ldots, m$, for otherwise $m = w(P) < m$. We now introduce a binary relation "$\to$" on $\{C_i \mid i = 1, 2, \ldots, m\}$ as follows: $C_i \to C_j$ if there are elements $x \in P_i$ and $y \in C_j - C_i$ such that $x > y$ in P. The definition is motivated by this observation:

if for some i, $C_i \not\to C_j$ for all j then $s(P) = w(P) - 1$.

To prove this let $x = \max(P_i)$, $C = \{y \in C_i \mid y \leqslant x\}$, and let $P' = P - C$. Then $w(P') = w(P) - 1$ and by the induction hypothesis there is a linear extension L' of P' consisting of a linear sum of $m - 1$ chains of P'. We claim $L = C \oplus L'$ is a linear extension of P; if not, there are elements $y \in C$ and $z \in P' \cap C_j$, for some $j \neq i$, with $y > z$. Hence $z < x$ and since $C_i \nrightarrow C_j$, it must be that $z \in C_i$; then $z \in C$, an impossibility.

We may therefore suppose that for each i there is some j such that $C_i \rightarrow C_j$. After suitable relabelling, there is a sequence $1, 2, \ldots, n$ of smallest length such that $C_1 \rightarrow C_2 \rightarrow \cdots \rightarrow C_n \rightarrow C_1$.

Choose $x_i \in P_i$ and $y_i \in C_i - C_{i-1}$ with $x_i > y_{i+1}$, for each $i = 1, 2, \ldots, n \pmod{n}$. Observe that $x_i > y_i$ for each i, $1 \leqslant i \leqslant n$. We conclude the proof by verifying that $\{x_1, y_1, x_2, y_2, \ldots, x_n, y_n\}$ must now contain an alternating cycle. Let us suppose that it is not itself a $2n$-cycle.

Case (i). Let $x_i > x_j$. Since $x_i \notin C_j$ there is some $x > x_j$ in C_j which is noncomparable with x_i. Further, since $y_{j+1} \notin C_j$ there is some $y < x_j$ in C_j which is noncomparable with y_{j+1}; then $\{x_i, x, y_{j+1}, y\}$ is a 4-cycle.

Case (ii). Let $y_i = y_j$, $i \neq j$. Then $C_{i-1} \rightarrow C_j$, contradicting the minimality of n.

Case (iii). Let $y_i < y_j$. If $y_i \notin C_j$ then there is $y < y_j$ in C_j noncomparable with y_i, so $\{x_{j-1}, x_j, y_i, y\}$ is a 4-cycle. If $y_i \in C_j$ then $C_{i-1} \rightarrow C_j$, again contradicting the minimality of n.

It follows that y_i is noncomparable with y_j for each $i \neq j$.

Case (iv). Let $x_i > y_j$, where $j \neq i$ and $j \neq i + 1$. Since y_j is noncomparable with y_i, $y_j \notin C_i$ so $C_i \rightarrow C_j$ which is again impossible.

Case (v). Let $x_i < y_j$. Then $y_i < y_j$ which was already ruled out.

This completes the proof.

An algorithm. Implicit in the proof of the theorem is an algorithm to construct a linear extension L of a cycle-free ordered set P which is optimal in the sense that $s_L(P) = s(P) = w(P) - 1$. The following procedure, though inductive, is based on a single covering $C_1, C_2, \ldots, C_{w(P)}$ of P by maximal chains.

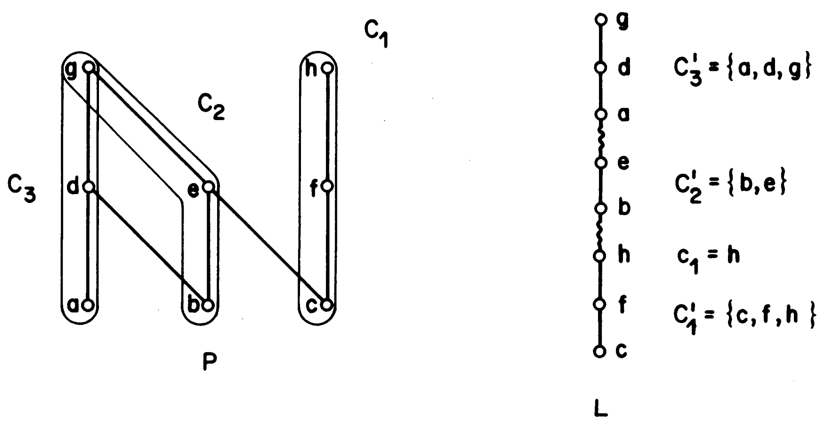


FIGURE 5

According to the proof of the theorem, in any such covering there is a chain (say, C_1) such that for any $i = 2, 3, \ldots, w(P)$, $C_1 \nrightarrow C_i$. Let $c_1 = \max(P_1)$, $C_1' = \{x \in C_1 \mid x \leq c_1\}$, and $Q = P - C_1'$. Then Q is covered by the chains $Q \cap C_2, \ldots, Q \cap C_{w(P)}$, and by inductive use of this algorithm Q has a linear extension

$$L' = C_2' \oplus C_3' \oplus \cdots \oplus C_{w(P)}'$$

with $s_{L'}(Q) = w(Q) - 1$, where $C_i' \subset C_i$ for each $i = 2, 3, \ldots, w(P)$. Then $L = C_1' \oplus L'$ is a linear extension of P for which $s_L(P) = w(Q) = w(P) - 1$ as required.

The algorithm is illustrated in Figure 5 for a particular cycle-free ordered set of width three.

References

1. G. Chaty, M. Chein, P. Martin and G. Petolla, *Some results about the number of jumps in circuit digraphs*, Proc. Fifth Southeastern Conf. Combinatorics, Graph Theory and Computing, Utilitas Math., Winnipeg, 1974, pp. 267–279.

2. M. Chein and P. Martin, *Sur le nombre de sauts d'une forêt*, C. R. Acad. Sci. Paris **275** (1972), 159–161.

3. M. Chein and H. Habib, *The jump number of dags and posets: an introduction*, Ann. Discrete Math. **9** (1980), 189–194.

4. O. Cogis and M. Habib, *Nombre de sauts et graphes série-parallèles*, RAIRO Inform. Théor. **13** (1979), 3–18.

5. R. P. Dilworth, *A decomposition theorem for partially ordered sets*, Ann. of Math. **51** (1950), 161–166.

6. J. Kuntzmann and A. Verdillon, *Recherche d'un ordre total minimal compatible avec un ordre partial donné*, Séminaire Institut de Mathématique de Grenoble, 1971.

7. W. R. Pulleyblank, *On minimizing setups in precedence constrained scheduling*, Discrete Appl. Math. (to appear).

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