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A NOTE ON TOPOLOGICAL HOM-FUNCTORS

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ABSTRACT. This note establishes internal criteria on a category C and a separator Σ in C which characterize the condition that the Σ-induced covariant hom-functor h_Σ: $C \to \mathrm{Set}$ is (epi, mono-source)-topological.

Introduction. Hoffmann [4] showed how topological functors (of Herrlich [2]) may be recovered from factorizations of sources and sinks in the domain category. This process was extended to (E, M)-topological functors in our paper [1], where we claimed, without elaborating details, that the results could be used to internally characterize the condition that a covariant hom-functor h_Σ: $C \to \mathrm{Set}$ is (epi, mono-source)-topological. In this note, we establish such a characterization which, in fact, depends on the supporting results of that paper. Our references are only intended to be immediately relevant rather than exhaustive, and our terminology is generally that of [1, 2, 3].

We note the following concepts before stating our main result:

DEFINITIONS. Let C be a category, Σ an object in C and e: $X \to Y$ a morphism in C. Further, let $h_\Sigma := C(\Sigma, -)$, the covariant hom-functor induced by Σ.

(1) C is said to *admit Σ-disjoint coproducts* iff for every coproduct sink $(u_i: X_i \to \coprod X_i)_I$ and morphism y: $\Sigma \to \coprod X_i$, there exist $i \in I$ and x: $\Sigma \to X_i$ such that $y = u_i \circ x$, i.e. iff $(h_\Sigma u_i)_I$ is an epi-sink.

(2) The morphism e: $X \to Y$ is said to be *Σ-coextendible* (cf. [3]) iff for every y: $\Sigma \to Y$, there exists x: $\Sigma \to X$ such that $y = e \circ x$, i.e. iff $h_\Sigma e$ is onto.

We have the following characterization:

THEOREM. *Let C be a category with an object Σ such that there exists at most a set of nonisomorphic objects C with $h_\Sigma C = \varnothing$. Then the following conditions are equivalent*:

(1) h_Σ: $C \to \mathrm{Set}$ *is an (epi, mono-source)-topological functor.*

(2) *C and Σ satisfy the following four conditions*:

(a) *C is a co-complete category*;

(b) *Σ is a separator*;

(c) *for any set S, every morphism $\Sigma \to^S \Sigma$ is a copower injection*;

(d) *C admits Σ-disjoint coproducts and every regular epi in C is Σ-coextendible.*

PROOF. (1) implies (2):

(2)(a) follows from the fact that Set is co-complete and h_Σ lifts colimits;

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(2)(b) follows from the fact that h_Σ is faithful;

(2)(c) follows from the fact that h_Σ has a left adjoint $L\colon S \to^S \Sigma$ such that the unit $\eta\colon \mathrm{Id} \to h_\Sigma L$ is pointwise onto; and

(2)(d) follows from the fact that h_Σ sends colimits to epi-sinks. The properties of h_Σ stated here may be verified from the results of Herrlich [**2**] and the well-known properties of faithful right adjoint hom-functors in [**3**].

(2) implies (1):

(2)(a) implies that $T := h_\Sigma$ is a right adjoint functor;

(2)(b) implies that T is faithful; and

(2)(c) implies that all front adjunctions for T are onto.

Now, let E be the class of all Σ-coextendible morphisms. We first show that C is an (E, M)-category (for the class M of T-initial mono-sources). To this end, we begin by noting that C is E-cowell-powered, since T is fibre-small and Set is cowell-powered. In view of [**1**, Theorem 1.3], it suffices to show that for each small source $(e_i\colon X \to X_i)_I$ with $e_i \in E (i \in I)$ and each morphism $f\colon X \to Y$, the morphism $e\colon Y \to Z$, in the following pushout diagram, belongs to E:

$$\begin{array}{ccc} X & \xrightarrow{e_i} & X_i \\ f\downarrow & & \downarrow q_i \\ Y & \xrightarrow[e]{} & Z \end{array} \qquad (i \in I).$$

FIGURE 1

This follows easily from 2(d), which implies that every colimit sink is sent to an epi-sink by T (cf. [**3**] for the construction of colimits from coequalizers and coproducts).

Now suppose that $(\eta, \varepsilon)\colon L \dashv T$, and let $(f_i\colon X \to TC_i)_I$ be a (small) mono-source in Set, where $C_i \in \operatorname{obj} C (i \in I)$. We note that, in view of (2c) and the fact that we may take $L\colon S \to^S \Sigma$, η is pointwise epic and, from the adjunction equation $T\varepsilon \circ \eta T = T$ and the definition of E, ε is pointwise in E. Now, let $\varepsilon C_i \circ Lf_i = m_i \circ u$ $(i \in I)$, where $u\colon LX \to C$ is in E and $(m_i)_I$ is in M. Then, $f_i = Tm_i \circ e$, where $e\colon X \to TC$ is given by $e = Tu \circ \eta X$, an epic. Hence e is an iso, and it only remains to show that $(m_i)_I$ is a T-initial source in order to conclude that T is an (epi, mono-source)-topological functor. To this end, let $(g_i\colon B \to C_i)$ be a source in C and $h\colon TB \to TC$ be a morphism with $Tg_i = Tm_i \circ h$ $(i \in I)$. Then as εB is in E and $(m_i)_I$ is in M, there exists a unique morphism $k\colon B \to C$ making the following diagram commute, i.e. easily, such that $Tk = h$ and $g_i = m_i \circ k$ $(i \in I)$, as required:

$$\begin{tikzcd}
LTB \arrow[rr, "\epsilon B"] \arrow[d, "Lh"'] & & B \arrow[dd, "g_i"] \arrow[ddll, dashed, "k"] \\
LTC \arrow[d, "\epsilon C"'] & & \\
C \arrow[rr, "m_i"'] & & C_i
\end{tikzcd}$$

FIGURE 2

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