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AN ANISOTROPIC LOCALLY HYPERBOLIC QUADRATIC SPACE

INTA BERTUCCIONI

ABSTRACT. We construct a semilocal normal domain A and a quadratic space over A, which is locally hyperbolic but anisotropic. This answers a question of H. Bass.

Let A be a commutative ring in which 2 is invertible. A quadratic space over A is a projective A-module M of finite rank, equipped with a regular symmetric bilinear form $\langle\ ,\ \rangle$. M is said to be hyperbolic if it is isometric to a quadratic space $H(P)$ of the form $P \oplus \operatorname{Hom}_A(P, A)$, where P is a projective A-module and $\langle p \oplus f, q \oplus g\rangle = f(q) + g(p)$. We say that M is locally hyperbolic if $M_{\mathfrak{m}}$ is isometric to $H(A_{\mathfrak{m}}^n)$ for every maximal ideal $\mathfrak{m}$ of A; and that it is isotropic if it contains a unimodular element u of length zero: $\langle u, u\rangle = 0$. In **[1]** Bass asked if a locally hyperbolic quadratic space over a semilocal domain A is always isotropic. He also gave an example showing that this is false if A has zerodivisors. We show the existence of a rank 4 quadratic space over a semilocal normal domain that is locally hyperbolic and anisotropic.

Let k be a field of characteristic $\neq 2$ and X, Y, Z, s, t, U, V, W indeterminates. Let B be the affine k-algebra $k[x, y, z] = k[X, Y, Z]/(Z^2 - (Y - X)(Y - X^2))$. B is nonsingular in codimension 1, hence normal by Macaulay's theorem. It is easy to check that there is an isomorphism $\phi\colon B[1/z] \xrightarrow{\sim} k[s, t, 1/f]$, where

$$f = s(t - s)(1 - t(t - s)),$$

given by $\phi(z) = (t(t - s) - 1)(t - s)^{-1}s^{-2}$, $\phi(x) = s\phi(z)$ and $\phi(y) = t\phi(z)$. This shows that $B[1/z]$ is factorial and by Nagata's theorem [3, Theorem 6.3] the divisor class group $\mathrm{Cl}(B)$ of B is generated by the classes of the prime ideals of height one that contain z, that is, by $\mathfrak{p} = Bz + B(y - x)$ and $\mathfrak{q} = Bz + B(y - x^2)$. From $\mathfrak{p}\mathfrak{q} = Bz \cdot (Bz + B(y - x) + B(y - x^2))$ and $\mathfrak{p}^2 = B(y - x) \cdot (B(y - x) + B(y - x^2))$, we conclude that $\mathrm{Cl}(B)$ is of order at most 2 and is generated by the class of $\mathfrak{p}$. Localizing B at $S = B \setminus \mathfrak{m}_1' \cup \mathfrak{m}_2'$, where $\mathfrak{m}_1' = Bz + Bx + By$ and $\mathfrak{m}_2' = Bz + B(x - 1) + B(y - 1)$, we get a semilocal normal ring A with 2 maximal ideals $\mathfrak{m}_1 = S^{-1}\mathfrak{m}_1'$ and $\mathfrak{m}_2 = S^{-1}\mathfrak{m}_2'$. Let C be the localization of

$$k[U, V, W]/(W^2 - UV) = k[u, v, w]$$

at the maximal ideal generated by u, v, w. Mapping x, y and z to u, $u + v(1 - u)$ and w respectively, we get an isomorphism of $A_{\mathfrak{m}_1}$ onto C. An isomorphism of $A_{\mathfrak{m}_2}$ onto C can be defined in a similar way. Hence $\mathrm{Cl}(A_{\mathfrak{m}_i})$ is the same as $\mathrm{Cl}(C)$, which

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is known to be the group of order 2 (see [**3**, p. 34 and Proposition 7.4]). Since $\mathrm{Cl}(A)$ maps surjectively onto $\mathrm{Cl}(A_{\mathfrak{m}})$, we conclude that $\mathrm{Cl}(A)$ is of order 2 and that $\mathrm{Cl}(A_{\mathfrak{m}})$ is generated by the class of $\mathfrak{p}A_{\mathfrak{m}}$. We now go back to A and consider the two A-modules $Q = A \oplus A$ and $P = \mathfrak{p} \oplus \mathfrak{p}$. The ring $A_{x(1-x)}$ is a Dedekind domain and $\mathfrak{p}^2_{x(1-x)}$ is principal, hence $P_{x(1-x)} \cong A_{x(1-x)} \oplus \mathfrak{p}^2_{x(1-x)} \cong Q_{x(1-x)}$. Patching P_x with Q_{1-x} over $A_{x(1-x)}$ by an isomorphism as above we get an A-module M. Clearly $\mathrm{End}_A M$ is locally isomorphic to a 2×2 matrix algebra and is, therefore, a quaternion algebra in the sense of [**2**]. The reduced norm defines on $N = \mathrm{End}_A M$ a structure of quadratic space over A. N is locally hyperbolic because $N_{\mathfrak{m}} \cong M_2(A_{\mathfrak{m}})$ with the quadratic form given by the determinant. We now show that N is anisotropic. Assume that it contains an isotropic unimodular element. Then it contains a hyperbolic plane $H(A)$ and it splits as $N = H(A) \perp N'$. The rank of M is 2 and its discriminant is trivial, hence $N' = H(A)$ and N is nothing but $M_2(A)$ with the quadratic form given by the determinant. By [**2**, Proposition 4.4] this implies that $\mathrm{End}_A M$ and $M_2(A)$ are isomorphic as A-algebras. By Morita duality this means that M is of the form $I \oplus I$ for some divisorial ideal I of A. Since $M_{1-x} = Q_{1-x}$, the ideal I_{1-x} is free. We have seen that $\mathrm{Cl}(A) \to \mathrm{Cl}(A_{1-x})$ is an isomorphism, hence I itself must be free. This is impossible because M_x is not free.

References

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Route de La Plaine 5, 1022 Chavannes, Switzerland