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CONULL HYPERSURFACES IN MINKOWSKI SPACE

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ABSTRACT. A submanifold of $\mathbf{M} = \mathrm{Gr}(2, \mathbf{C}^4)$ is conull when its conormal space is in the kernel of the dualized conformal metric of $\mathbf{M}$. We show that there are no conull compact complex 3-dimensional submanifolds of $\mathbf{M}$.

Let $\mathbf{M}$ be complex Minkowski space, complex analytically homeomorphic to $\mathrm{Gr}(2, \mathbf{C}^4)$, the Grassmannian of 2-planes in $\mathbf{C}^4$.

Each linear $\mathbf{P}^2$ in $\mathbf{M}$ is a null manifold i.e. the tangent space $T_x(\mathbf{P}^2)$ consists entirely of null vectors, for each $x \in \mathbf{P}^2$ cf. [**4**, 5.1].[2] Of course, any curve lying in such a $\mathbf{P}^2$ is null, but there are other null curves.

EXAMPLE 1.

$$t \mapsto \begin{pmatrix} 1 & t & t^2 & t^4 \\ 0 & 1 & 2t & 4t^3 \end{pmatrix}$$

is a null smooth $\mathbf{P}^1$ contained in $\mathbf{M}$, but is not contained in any hyperplane section via the Plucker embedding $\mathbf{M} \hookrightarrow \mathbf{P}^5$. (See Griffiths and Harris [**2**, 2.4] for a local description of null curves in Grassmannians.)

Let X be a complex submanifold of M. We say that X is conull if the conormal space of X, $N^*X \subset T^*\mathbf{M}$, consists entirely of null covectors. It is an exercise in linear algebra to verify that for $\dim(X) = 2$, X is null precisely when X is conull. However, the plethora of null curves is in marked contrast to the lack of conull hypersurfaces.

THEOREM. *Let X be a smooth compact complex hypersurface of* $\mathbf{M}$. *Then X is not conull.*

PROOF. Let $G = \mathrm{PGL}(3, \mathbf{C})$ be the projective linear group. The group G acts transitively on $\mathbf{M}$, and defines an action on the cotangent bundle, $T^*\mathbf{M}$. This latter action has 3 orbits—the zero section, the null covectors, and the open orbit. It is well known that the normal bundle, NX, of X in $\mathbf{M}$ is ample (e.g. [**3**, 2.9]). By [**1**, 4.5.2], N^*X must meet the open orbit. ■

The theorem is not a local result.

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[2] The proof of the converse statement, as pointed out to me by Robert Bryant, is a straightforward local calculation: A null surface in $\mathbf{M}$ is necessarily a linear $\mathbf{P}^2$.

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EXAMPLE 2. Let Y be the smooth points of a singular hyperplane section of $\mathbf{M}$. Then Y is conull.

REMARK. The theorem is true with $\mathbf{M}$ being any Grassmannian or quadric, and should remain valid with $\mathbf{M}$ replaced by G/P, G a simple complex Lie group and P a maximal parabolic subgroup. A smooth hypersurface of G/P does have an ample normal bundle, but I do not know, in general, whether G/P is rigged, in the sense of [**1**, §3].

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