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ON THE UNIFORM ASYMPTOTIC STABILITY IN FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. We consider a system of functional differential equations $x'(t) = F(t, x_t)$ and obtain conditions on a Liapunov functional to insure the uniform asymptotic stability of the zero solution.

1. Introduction. Following the work of Yoshizawa [2], Burton [1] obtained sufficient conditions of the uniform asymptotic stability in the retarded functional differential equation $x'(t) = F(t, x_t)$ on a Liapunov functional. He showed that it is not necessary to require $F(t, x_t)$ bounded for x_t bounded. Now we use the Razumikhin condition so that it is not necessary to require $V'(t, x_t) \leqslant -W(|x(t)|)$ for all $t \geqslant 0$. This work generalized Burton's result.

For $x \in R^n$, let $|x|$ be $\max_{1 \leqslant i \leqslant n}|x_i|$. Given $h > 0$, let C denote the space of continuous functions from $[-h, 0]$ into R^n and for $\phi \in C$, $\|\phi\| = \sup_{-h \leqslant \theta \leqslant 0}|\phi(\theta)|$. For $\phi \in C_H = \{\phi\colon \phi \in C, \|\phi\| \leqslant H\}$, let

$$|||\phi||| = \left(\sum_{i=1}^{n} \int_{-h}^{0} \phi_i^2(s)\, ds \right)^{1/2},$$

where ϕ_i are the components of ϕ.

For $t_0 \in R$, $A > 0$, $t \in [t_0, t_0 + A)$ and a continuous function x from $[t_0 - h, t_0 + A]$ into R^n, let $x_t \in C$ be defined by $x_t(\theta) = x(t + \theta)$, $\theta \in [-h, 0]$.

2. Uniform asymptotic stability.

LEMMA. *Let F be a family of continuous functions f: $[a, b] \to [0, 1]$ and W: $[0, \infty) \to [0, \infty)$ be a continuous nondecreasing function, and $W(s) > 0$ if $s > 0$. If there exists $\alpha > 0$ with $\int_a^b f(t)\, dt \geqslant \alpha$ for any $f \in F$ then there exists $\beta > 0$ with $\int_0^1 W(f(t))\, dt \geqslant \beta$.*

PROOF. For any $f \in F$, let $E = \{t\colon f(t) \geqslant \alpha/2(b - a), a \leqslant t \leqslant b\}$ and $m(E)$ be the measure of E. If $m(E) < \alpha/2$, then

$$\alpha \leqslant \int_a^b f(t)\, dt = \int_E f(t)\, dt + \int_{[a,b]-E} f(t)\, dt < \alpha/2 + \alpha/2 = \alpha,$$

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a contradiction. Hence $m(E) \geqslant \alpha/2$ and

$$\int_a^b W(f(t))\,dt \geqslant \int_E W(f(t))\,dt \geqslant \int_E W(\alpha/2(b-a))\,dt \geqslant W(\alpha/2(b-a))\alpha/2 \overset{\text{def}}{=} \beta.$$

This completes the proof.

We consider the retarded functional differential equation

$$x'(t) = F(t, x_t), \tag{1}$$

where $x'(t)$ is the right-hand derivative of $x(t)$ and $F(t, x_t)$ a continuous function from $R \times C_H$ into R^n, $F(t, 0) = 0$. For continuation of solution, we suppose that F takes closed bounded sets of $R \times C_H$ into closed bounded sets of R^n.

Denote by $x(t_0, \phi)$ a solution of (1) with initial condition $\phi \in C_H$ where $x_{t_0}(t_0, \phi) = \phi$ and we denote by $x(t) = x(t, t_0, \phi)$ the value of $x(t_0, \phi)$ at t.

Let $V(t, \phi)$ be a continuous nonnegative functional defined in $[0, \infty) \times C_H$. The upper right-hand derivative of V along solution of (1) is defined to be

$$V'(t, x_t(t_0, \phi)) = \overline{\lim_{\delta \to 0^+}} \{V(t+\delta, x_{t+\delta}(t_0, \phi)) - V(t, x_t(t_0, \phi))\}/\delta.$$

We suppose that $V'(t, x_t)$ exists.

Let W_1, W_2, W_3, W be continuous nondecreasing functions and P be a continuous function from $[0, \infty)$ into $[0, \infty)$ with $W_i(r) > 0$, $W(r) > 0$, $P(r) > r$ if $r > 0$ and $W_i(0) = 0$.

THEOREM. *Suppose there are functions W_1, W_2, W_3, W, P, V as above, which also satisfy the following conditions*:

(i) $W_1(|\Psi(0)|) \leqslant V(t, \Psi) \leqslant W_2(|\Psi(0)|) + W_3(|||\Psi|||)$ *for any* $\Psi \in C_H$.

(ii) *For any* $t_0 \geqslant 0$ *and any* $\phi \in C_H$

$$V'(t, x_t(t_0, \phi)) < 0 \quad \textit{if } V(t, x_t(t_0, \phi)) = W_2(\|\phi\|) + W_3(|||\phi|||) \quad (t_0 \leqslant t \leqslant t_0 + h),$$

and

$$V'(t, x_t(t_0, \phi)) \leqslant -W(|x(t, t_0, \phi)|) \quad \textit{if } P(V(t, x_t(t_0, \phi))) > V(\xi, x_\xi(t_0, \phi))$$
$$(t \geqslant t_0 + h;\ t - h \leqslant \xi \leqslant t).$$

Then the zero solution of (1) *is uniformly asymptotically stable.*

PROOF. We first prove the uniform stability. Given $\varepsilon > 0$ ($\varepsilon < H$, $W_1(\varepsilon) < H$), choose $\delta > 0$ such that $\delta < \varepsilon$, $W_2(\delta) < W_1(\varepsilon)/2$, and $W_3(\delta\sqrt{nh}) < W_1(\varepsilon)/2$. Let $t_0 \geqslant 0$ and $\|\phi\| < \delta$. We shall show that

$$V(t, x_t(t_0, \phi)) < W_1(\varepsilon) \qquad (t \geqslant t_0). \tag{2}$$

Obviously,

$$V(t_0, \phi) \leqslant W_2(|\phi(0)|) + W_3(|||\phi|||) \leqslant W_2(\delta) + W_3(\delta\sqrt{nh}) < W_1(\varepsilon).$$

For each $t \in [t_0, t_0 + h)$, if $V(t, x_t) < W_2(\|\phi\|) + W_3(|||\phi|||)$, then $V(t, x_t) < W_1(\varepsilon)$, if $V(t, x_t) = W_2(\|\phi\|) + W_3(|||\phi|||)$, from condition (ii) we get $V(t + \Delta t, x_{t+\Delta t}) \leqslant W_2(\|\phi\|) + W_3(|||\phi|||)$ for all sufficiently small $\Delta t > 0$. It implies that $V(t, x_t) < W_1(\varepsilon)$

for all $t \in [t_0, t_0 + h)$. Thus, if (2) fails, then there exists $t_1 \geqslant t_0 + h$ such that

$$V(t_1, x_{t_1}) = W_1(\varepsilon), \quad V(t, x_t) \leqslant W_1(\varepsilon) \qquad (t \leqslant t_1).$$

Let $d = \inf_{W_2(\|\phi\|) + W_3(|||\phi|||) < r < W_1(\varepsilon)} [P(r) - r]$. Obviously, there exists $T \in (t_0 + h, t_1)$ such that

(a) $W_2(\|\phi\|) + W_3(|||\phi|||) \leqslant W_1(\varepsilon) - \frac{1}{e} d < V(T, x_T) < W_1(\varepsilon)$, where $e > 1$,

(b) $V'(T, x_T) > 0$.

From (a),

$$P(V(T, x_T)) \geqslant V(T, x_T) + d > W_1(\varepsilon) + \left(1 - \frac{1}{e}\right) d > V(\xi, x_\xi) \qquad (t_0 \leqslant \xi \leqslant T).$$

From condition (ii), we have $V'(T, x_T) \leqslant -W(|x(T)|) \leqslant 0$, which contradicts (b). Hence, (2) holds.

By (2) and condition (i), we get $|x(t)| < \varepsilon$ for $t \geqslant t_0$. Since δ is independent of t_0, this proves the uniform stability.

Next, we prove the uniform asymptotic stability. For $H^* = \min[H, 1]$ choose $\delta > 0$ such that $|x(t, t_0, \phi)| < H^*$ for $t \geqslant t_0$, if $t_0 \geqslant 0$ and $\|\phi\| \leqslant \delta$. From condition (i), we have

$$V(t, x_t(t_0, \phi)) \leqslant W_2(H^*) + W_3\left(H^* \sqrt{nh}\right).$$

Choose a positive $B > W_2(H^*) + W_3(H^* \sqrt{nh})$. For given $\varepsilon > 0$ ($\varepsilon < H$), let $\bar{d} = \inf_{W_1(\varepsilon) \leqslant r \leqslant B} (P(r) - r)$, and N be a positive integer satisfying $W_1(\varepsilon) + (N-1)\bar{d} < B \leqslant W_1(\varepsilon) + N\bar{d}$. We shall show that there exists $T_1 > t_0 + h$ such that

$$V(T_1, x_{T_1}(t_0, \phi)) < W_1(\varepsilon) + (N-1)\bar{d}. \tag{3}$$

If not, then

$$V(t, x_t) \geqslant W_1(\varepsilon) + (N-1)\bar{d} \qquad (t \geqslant t_0 + h),$$

and

$$P(V(t, x_t)) \geqslant V(t, x_t) + \bar{d} \geqslant W_1(\varepsilon) + N\bar{d} \geqslant B > V(\xi, x_\xi) \qquad (t_0 \leqslant \xi \leqslant t).$$

From (ii) we have $V'(t, x_t) \leqslant -W(|x(t)|)$ $(t \geqslant t_0 + h)$; it follows that

$$V(t, x_t) < B - \int_{t_0+h}^{t} W(|x(s)|)\, ds. \tag{4}$$

If $V(t, x_t) \geqslant W_1(\varepsilon)$, then

$$W_2(|x(t)|) + W_3(|||x_t|||) > V(t, x_t) > W_1(\varepsilon).$$

Therefore, either $W_2(|x(t)|) \geqslant W_1(\varepsilon)/2$ or $W_3(|||x_t|||) \geqslant W_1(\varepsilon)/2$. Let $E_1 = \{t: W_3(|||x_t|||) \geqslant W_1(\varepsilon)/2, t \geqslant t_0\}$ and $E_2 = [t_0, \infty) - E_1$. If $t \in E_1$, then there exists a constant $a > 0$ with $|||x_t||| > a$. If $t \in E_2$, then there exists a constant $b > 0$ with $|x(t)| > b$. In case $t \in E_1$, we have

$$\sum_{i=1}^{n} \int_{-h}^{0} x_i^2(t + \theta)\, d\theta \geqslant a^2,$$

then

$$\int_{t-h}^{t} \frac{1}{n} \sum_{i=1}^{n} x_i^2(s)\, ds \geqslant \frac{a^2}{n} \stackrel{\text{def}}{=} \alpha.$$

Since $|x(t)| < 1$, we have

$$|x(t)| = \max_i |x_i(t)| \geqslant \frac{1}{n} \sum_{i=1}^{n} x_i^2(t).$$

Then from the Lemma, there exists $\beta \geqslant 0$ such that

(5) $$\int_{t-h}^{t} W(|x(s)|)\, ds \geqslant \int_{t-h}^{t} W\left(\frac{1}{n} \sum_{i=1}^{n} x_i^2(s)\right) ds \geqslant \beta.$$

Let K be the positive integer satisfying $K > B \geqslant (K-1)$ and $T_1 = t_0 + (K+1)h + 2B/W(b)$, we have either

($\bar{a}$) $m(E_1 \cap [t_0 + h, T_1]) \geqslant Kh$ or

($\bar{b}$) $m(E_2 \cap [t_0 + h, T_1]) \geqslant 2B/W(b)$.

If ($\bar{a}$) holds, then in $E_1 \cap [t_0 + h, T_1]$ there exist K points $t_1 < t_2 < \cdots < t_k$ satisfying $t_1 \geqslant t_0 + 2h$ and $t_j - t_{j-1} \geqslant h$ $(j = 2, 3, \ldots, K)$. From (4) and (5), we have

$$\begin{aligned} V(T_1, x_{T_1}) &< B - \int_{t_0+h}^{T_1} W(|x(s)|)\, ds \\ &\leqslant B - \sum_{j=1}^{k} \int_{t_j-h}^{t_j} W\left(\frac{1}{n} \sum_{i=1}^{n} x_i^2(s)\right) ds \leqslant B - k\beta < 0. \end{aligned}$$

If ($\bar{b}$) holds, from (4) we have

$$V(T_1, x_{T_1}) < B - \int_{E_2 \cap [t_0+h, T_1]} W(b)\, ds = B - W(b) m(E_2 \cap [t_0 + h, T_1]) < 0.$$

Thus either ($\bar{a}$) or ($\bar{b}$) implies $V(T_1, X_{T_1}) < 0$, a contradiction to $V(t, x_t) \geqslant 0$. Hence (3) holds.

In the following, we will show that

(6) $$V(t, x_t(t_0, \varphi)) < W_1(\varepsilon) + (N-1)\bar{d} \quad \text{for all } t \geqslant T_1.$$

If (6) is not true, then there exists $\sigma > T_1$ such that $V(\sigma, x_\sigma) \leqslant W_1(\varepsilon) + (N-1)\bar{d}$ and

(A) $B - W_2(H^*) - W_3(H^*\sqrt{nh}) > W_1(\varepsilon) + (N-1)\bar{d} - V(\sigma, x_\sigma)$,

(B) $V'(\sigma, x_\sigma) > 0$.

From (A), we get

$$\begin{aligned} P(V(\sigma, x_\sigma)) &\geqslant V(\sigma, x_\sigma) + \bar{d} \\ &> W_1(\varepsilon) + (N-1)\bar{d} - B + W_2(H^*) + W_3(H^*\sqrt{nh}) + \bar{d} \\ &= W_1(\varepsilon) + N\bar{d} - B + W_2(H^*) + W_3(H^*\sqrt{nh}) \\ &\geqslant W_2(H^*) + W_3(H^*\sqrt{nh}) \geqslant V(\xi, x_\xi) \qquad (t_0 \leqslant \xi \leqslant \sigma). \end{aligned}$$

From condition (ii) we have $V'(\sigma, x_\sigma) \leqslant -W(|x(\sigma)|) \leqslant 0$, which contradicts (B). Therefore, (6) holds.

Similarly, there exists $T_2, T_3, \ldots, T_N$ such that

$$V(t, x_t(t_0, \phi)) < W_1(\varepsilon) + (N-k)\bar{d} \quad \text{for } t \geqslant T_k,\ k = 2, 3, \ldots, N.$$

Then $V(t, x_t(t_0, \phi)) < W_1(\varepsilon)$ for all $t \geqslant T_N$. From condition (i) we have $|x(t)| < \varepsilon$ for all $t \geqslant T_N$, where

$$T_N = t_0 + N((k+1)h + 2B/W(b)).$$

Since $N((k+1)h + 2B/W(b))$ is independent of t_0, we have completed the proof of the theorem.

Example. Consider the equation

$$x'(t) = -a(t)x(t) + b(t)x(t-h) \tag{7}$$

where $a(t)$ and $b(t)$ are continuous functions, $0 < a \leqslant a(t) < \infty$, $|b(t)| \leqslant b < \mu a$, $0 < \mu < 1$.

One can choose $V(t, x_t) = \frac{1}{2}x^2(t)$, $W_1(|x(t)|) = \frac{1}{4}x^2(t)$, $W_2(|x(t)|) = x^2(t)$, $W_3(|||x_t|||) = |||x_t|||^2$ and $P(s) = qs$, $q > 1$.

For $t \in [t_0, t_0 + h)$, if $V(t, x_t) = W_2(\|\phi\|) + W_3(|||\phi|||)$, that is $\frac{1}{2}x^2(t) = \|\phi\|^2 + |||\phi|||^2$. Then

$$\begin{aligned} V'(t, x_t) &= x(t)x'(t) = -a(t)x^2(t) + b(t)x(t)x(t-h) \\ &\leqslant -ax^2(t) + \frac{b}{2}\left[x^2(t) + x^2(t-h)\right] \\ &\leqslant -\left(a - \frac{b}{2}\right)x^2(t) + \frac{b}{2}\|\phi\|^2 = -\left(2a - \frac{3b}{2}\right)\|\phi\|^2 - (2a-b)|||\phi|||^2 \\ &< 0. \end{aligned}$$

For $t \in [t_0 + h, \infty)$ if $P(V(t, x_t)) > V(\xi, x_\xi)$ $(t - h \leqslant \xi \leqslant t)$, that is $qx^2(t) > x^2(\xi)$ $(t - h \leqslant \xi \leqslant t)$, then $qx^2(t) > x^2(t-h)$.

$$\begin{aligned} V'(t, x_t) &\leqslant -\left(a - \frac{b}{2}\right)x^2(t) + \frac{b}{2}x^2(t-h) \\ &\leqslant -\left(a - \frac{b}{2}\right)x^2(t) + \frac{b}{2}qx^2(t) = -\left(a - b\left(\frac{1+q}{2}\right)\right)x^2(t). \end{aligned}$$

If we choose $q = 2/\mu - 1$, then $a - b((1+q)/2) > 0$. Let

$$W(|x(t)|) = (a - b((1+q)/2))x^2(t).$$

We can see that the conditions of the Theorem are satisfied. Therefore, the zero solution of (7) is uniformly asymptotically stable.

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