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ANOTHER CHARACTERIZATION OF BLO

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ABSTRACT. It is shown that a locally integrable function f on $\mathbf{R}^n$ has bounded lower oscillation ($f \in \mathrm{BLO}$) if and only if $f = MF + h$, where F has bounded mean oscillation ($F \in \mathrm{BMO}$) and $MF < \infty$ a.e., and h is bounded. Here, MF is a variant of the familiar Hardy-Littlewood maximal function: $MF = \sup_{Q \ni x} Q(F)$ (no absolute values), where $Q(F)$ is the mean value of F over the cube Q.

We consider real-valued locally integrable functions f on $\mathbf{R}^n$. When Q is a cube in $\mathbf{R}^n$ with sides parallel to the coordinate axes, we denote by $Q(f)$ the mean value of f over Q. Define a maximal function Mf of f by

$$(Mf)(x) = \sup_{Q \ni x} Q(f) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q f(y)\, dy \qquad (x \in \mathbf{R}^n),$$

where the supremum extends over all cubes Q that contain x. Note that f is not assumed to be nonnegative so that Mf may take on negative values. However, it follows from the differentiation theorem that $Mf \geq f$ a.e. Moreover, it is clear $|Mf|$ is dominated by $M|f|$, the latter being the familiar Hardy-Littlewood maximal function of f, and so, in particular, the maximal operator M is bounded on L^2. We shall make use of the obvious fact that if Q and Q' are two cubes with $Q \subset Q'$, then $Q'(f)$ is dominated by $(Mf)(x)$ for every x in Q, and so

$$Q'(f) \leq \inf_Q Mf \qquad (Q \subset Q'). \tag{1}$$

A locally integrable function f on $\mathbf{R}^n$ is said to be of *bounded mean oscillation* ($f \in \mathrm{BMO}$) if

$$\|f\|_{\mathrm{BMO}} = \sup_Q Q(|f - Q(f)|) \tag{2}$$

is finite. The space BMO is a linear space and, when the constant functions are factored out, the functional in (2) is a norm under which BMO is a Banach space. It is well known (cf. [3, p. 227]) that replacing (2) by the corresponding quadratic functional results in an equivalent norm on BMO. In particular, there is a constant c, depending only on the dimension n, such that

$$\left(\frac{1}{|Q|} \int_Q |f(x) - Q(f)|^2\, dx \right)^{1/2} \leq c\|f\|_{\mathrm{BMO}}, \tag{3}$$

for every cube Q in $\mathbf{R}^n$.

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Recently, R. R. Coifman and R. Rochberg [2] have introduced the space BLO of functions of *bounded lower oscillation*. The space is defined analogously to BMO except that in (2) one subtracts from f the essential infimum

$$f_Q = \operatorname*{ess\,inf}_{x \in Q} f(x)$$

instead of the mean value $Q(f)$. Thus, a locally integrable function f on $\mathbf{R}^n$ is in BLO if

$$\|f\|_{\mathrm{BLO}} = \sup_Q \left(Q(f) - f_Q\right) \tag{4}$$

is finite. The BLO property need not be preserved under multiplication by negative constants so, in particular, BLO is not a linear space, and, despite the notation, the functional in (4) is not a norm. It is easy to check, however, that $\|\cdot\|_{\mathrm{BLO}}$ is subadditive and positive-homogeneous.

Note that every L^∞-function is in BLO, with

$$\|f\|_{\mathrm{BLO}} \leq 2\|f\|_{L^\infty} \qquad (f \in L^\infty), \tag{5}$$

and every BLO-function is in BMO, with

$$\|f\|_{\mathrm{BMO}} \leq 2\|f\|_{\mathrm{BLO}} \qquad (f \in \mathrm{BLO}). \tag{6}$$

Using a representation theorem of Carleson for BMO-functions, Coifman and Rochberg [2] have shown that every BMO-function is representable as the difference of two BLO-functions. Moreover, they have shown that BLO-functions arise, modulo bounded functions, as logarithms of maximal functions. In other words, a locally integrable function f belongs to BLO if and only if

$$f = \alpha \log MF + h, \tag{7}$$

where α is a nonnegative constant, F is a nonnegative locally integrable function whose maximal function MF is finite a.e., and h is an L^∞-function.

Let us note also the result of [1] that if $f \in$ BMO (and has finite maximal function), then the decreasing rearrangement f^* (of $|f|$) is in BLO; in fact, this condition characterizes the rearrangements of BMO-functions.

In this note, we shall obtain a different description of BLO-functions. We shall show that they arise, modulo bounded functions, as the maximal functions MF of BMO-functions F. Of course, we must exclude those F for which MF is identically infinite.

We shall need two lemmas, the first of which may be regarded as a refinement of Theorem 4.2 in [1].

LEMMA 1. *If $F \in BMO$ and if Q is any cube in $\mathbf{R}^n$, then*

$$Q(MF) \leq c\|F\|_{BMO} + \inf_Q MF, \tag{8}$$

where c is a constant depending on the dimension n. In particular, if MF is not identically infinite, then $MF \in BLO$ and

$$\|MF\|_{BLO} \leq c\|F\|_{BMO}. \tag{9}$$

Proof. Fix Q and let $\bar{Q}$ be the cube concentric to Q with dimensions three times as large. Write

$$F = (F - \bar{Q}(F))\chi_{\bar{Q}} + \left[\bar{Q}(F)\chi_{\bar{Q}} + F\chi_{\bar{Q}^c}\right] = G + H,$$

say.

By the Cauchy-Schwarz inequality,

$$Q(MG) \leqslant \left(\frac{1}{|Q|}\int_Q (MG)^2\,dx\right)^{1/2} \leqslant |Q|^{-1/2}\|MG\|_{L^2(\mathbf{R}^n)}.$$

But M is bounded on L^2, so using the definition of G, the fact that $|\bar{Q}| = 3^n|Q|$, and (3), we obtain

$$Q(MG) \leqslant c\|F\|_{\mathrm{BMO}}. \tag{10}$$

Next we shall show that

$$Q(MH) \leqslant c\|F\|_{\mathrm{BMO}} + \inf_Q MF, \tag{11}$$

which, together with (10) and the fact that $F = G + H$, will establish the desired result (8). In order to establish (11), it will suffice to show that $MH(x)$ is dominated by the right-side of (11) for every $x \in Q$, and for this it will be enough to show that

$$P(H) \leqslant c\|F\|_{\mathrm{BMO}} + \inf_Q MF \tag{12}$$

for every cube P in $\mathbf{R}^n$ containing x.

The result follows directly from (1) if P does not meet $\bar{Q}^c$, for then $H = \bar{Q}(F)$ on P and so $P(H) = \bar{Q}(F) \leqslant \inf_Q MF$. So suppose $P \cap \bar{Q}^c \neq \varnothing$, and let P' be the smallest cube containing both P and $\bar{Q}$. Since P contains the point x of Q, it is clear that

$$|P'| \leqslant c|P| \tag{13}$$

for some constant c depending only on the dimension n. Furthermore, from the definition of H,

$$\int_P (H - P'(F)) \leqslant \int_{P'} |H - P'(F)| = \int_{\bar{Q}} |\bar{Q}(F) - P'(F)| + \int_{P' \cap \bar{Q}^c} |F - P'(F)|$$
$$\leqslant \left(\int_{\bar{Q}} + \int_{P' \cap \bar{Q}^c}\right)|F - P'(F)| = \int_{P'} |F - P'(F)|,$$

so, using (2) and (13), we obtain $P(H - P'(F)) \leqslant c\|F\|_{\mathrm{BMO}}$. Since $P' \supset Q$, an appeal to (1) now gives

$$P(H) = P(H - P'(F)) + P'(F) \leqslant c\|F\|_{\mathrm{BMO}} + \inf_Q MF.$$

This establishes (12) and hence, as we remarked above, completes the proof of (8).

It is now clear from (8) that if $(MF)(x)$ is finite at any one point x in $\mathbf{R}^n$, then $(MF)(y)$ is finite a.e. on every cube Q containing x, hence on all of $\mathbf{R}^n$. In that case, we may subtract $\inf_Q MF$ from each side of (8) and take the supremum over all Q to obtain (9).

We require one further lemma.

LEMMA 2. *A locally integrable function f on $\mathbf{R}^n$ belongs to BLO if and only if $Mf - f$ belongs to L^∞. Furthermore,*

$$\|Mf - f\|_{L^\infty} = \|f\|_{BLO}. \tag{14}$$

PROOF. Suppose first that f belongs to BLO. Let x be any Lebesgue point of f and let Q be any cube containing x. Then $f(x) \geqslant f_Q$ and so

$$Q(f) - f(x) \leqslant Q(f) - f_Q \leqslant \|f\|_{\mathrm{BLO}}.$$

Taking the supremum over all cubes Q containing x, and then the supremum over all Lebesgue points x of f, we find that $Mf - f$ belongs to L^∞ and

$$\|Mf - f\|_{L^\infty} \leqslant \|f\|_{\mathrm{BLO}}. \tag{15}$$

Conversely, suppose $Mf - f$ belongs to L^∞ and let Q be any cube in $\mathbf{R}^n$. Any point x in Q for which

$$f(x) < Q(f) - \|Mf - f\|_{L^\infty} \tag{16}$$

must satisfy

$$(Mf)(x) - f(x) \geqslant Q(f) - f(x) > \|Mf - f\|_{L^\infty},$$

and consequently such points x constitute a set of measure zero. Hence the essential infimum f_Q is at least as large as the value on the right-hand side of (16), and so

$$Q(f) - f_Q \leqslant \|Mf - f\|_{L^\infty}.$$

Taking the supremum over all Q, we obtain the reverse inequality to (15), and hence (14) is established.

Our main result is as follows.

THEOREM. *A locally integrable function f on $\mathbf{R}^n$ belongs to BLO if and only if there are functions h in L^∞ and F in BMO with MF finite a.e. such that*

$$f = MF + h. \tag{17}$$

Furthermore,

$$\|f\|_{BLO} \sim \inf(\|F\|_{BMO} + \|h\|_{L^\infty}), \tag{18}$$

where the infimum extends over all representations of the form (17).

PROOF. If f has a representation as in (17), then MF belongs to BLO by virtue of Lemma 1. Since h is bounded, hence in BLO, we see that f is in BLO. Furthermore, from (5) and (9),

$$\|f\|_{\mathrm{BLO}} \leqslant \|MF\|_{\mathrm{BLO}} + \|h\|_{\mathrm{BLO}} \leqslant c\|F\|_{\mathrm{BMO}} + 2\|h\|_{L^\infty},$$

so taking the infimum over all representations (17) of f, we obtain

$$\|f\|_{\mathrm{BLO}} \leqslant c \inf(\|F\|_{\mathrm{BMO}} + \|h\|_{L^\infty}). \tag{19}$$

Conversely, if $f \in \mathrm{BLO}$, then Mf is finite a.e., by Lemma 2. Furthermore, by the same lemma, the function $f - Mf$ is bounded, and so $f = Mf + (f - Mf)$ is a representation of the form (17) with $F = f$ and $h = f - Mf$. Moreover, from (6) and (14),

$$\|F\|_{\mathrm{BMO}} + \|h\|_{L^\infty} = \|f\|_{\mathrm{BMO}} + \|f - Mf\|_{L^\infty} \leqslant 3\|f\|_{\mathrm{BLO}}$$

so, combining this observation with (19), we establish (18).

REMARKS. The above analysis can also be carried through for a cube Q_0 instead of all of $\mathbf{R}^n$. In this case, BLO-functions on Q_0 are bounded below and so, by addition of suitable constants, can be rendered nonnegative. In this case, one obtains a result of the form (17) with M the usual Hardy-Littlewood maximal function $\sup_Q Q(|f|)$.

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