PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 85, Number 4, August 1982

ON THE MONODROMY OF HIGHER LOGARITHMS

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ABSTRACT. The (multivalued) higher logarithms are interpreted, by studying their monodromy, as giving well-defined maps from $\mathbf{P}^1_{\mathbf{C}} - \{3 \text{ points}\}$ into certain complex nilmanifolds with $\mathbf{C}^*$-actions.

The purpose of this note is to exhibit a family of unipotent representations of $\mathbf{Z} * \mathbf{Z}$ arising naturally from the monodromy of the higher logarithms $\ln_k$ (see [4]), and thereby interpret each $\ln_k$ as yielding a well-defined map ρ_k of $\mathbf{P}^1_{\mathbf{C}} - \{0, 1, \infty\}$ into a $(k+1)$-dimensional complex nilmanifold M_{k+1} equipped with a $\mathbf{C}^*$-action. Moreover, a natural holomorphic connection ∇_k is shown to exist on each fibration $M_{k+1} \to M_{k+1}/\mathbf{C}^*$, with respect to which ρ_k is flat. The role of dilogarithm in the study of volumes of hyperbolic 3-manifolds [5], arithmetic [1, 2], K-theory and Kac-Moody Lie algebras [3] leads one to hope for such interesting links in the case of higher logarithms as well. Furthermore, the tower of nilmanifolds associated to $\mathbf{P}^1_{\mathbf{C}} - \{3 \text{ points}\}$ via $\{\ln_k\}$ suggests, following a remark of P. Deligne, relations to Sullivan's theory of differential forms. We hope to pursue this at some future time. Finally, we have come to learn recently of an independent and very elegant construction of these nilmanifolds due to J. W. Milnor (unpublished).

We would like to thank Spencer Bloch for his constant and friendly encouragement.

For x in $\mathbf{C}$, set $\ln_0(x) = x/(1-x)$ and $L(x) = (2\pi i)^{-1}\log(x)$. Then $\ln_k$ is defined recursively by

$$\ln_k(x) = \int_0^x \ln_{k-1}(t)\, dL(t).$$

It follows easily that each $\ln_k$ satisfies the $(k+1)$st order, homogeneous, algebraic differential equation

$$(*) \qquad \frac{d}{dx}\left((1-x)\frac{d}{dx}\left(\left(x\frac{d}{dx}\right)^{k-1} U\right)\right) = 0.$$

This equation has regular singular points at 1 and ∞ if $k = 1$, and at 0, 1, and ∞ if $k > 1$.

Received by the editors August 3, 1981.
1980 *Mathematics Subject Classification*. Primary 53C30, 22E40; Secondary 33A75.

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0002-9939/82/0000-1081/$02.00

Now let, for each integer $m > 2$,

$$N_m = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\} \subset \mathrm{GL}_m, \qquad U_m = \left\{ \left(\begin{array}{c|c} 1 & * \\ \hline 0 & I \end{array} \right) \right\},$$

and let $[\ \ ]_m$ denote the one-parameter subgroup $\mathbf{G}_a \to N_m$ given by

$$y \mapsto \begin{bmatrix} 1 & 0 & & & & \cdots & 0 \\ & 1 & y & y^2/2! & y^3/3! & \cdots & y^{(m-1)}/(m-1)! \\ & & 1 & y & y^2/2! & \cdots & y^{(m-2)}/(m-2)! \\ & & & 1 & & \ddots & \vdots \\ & & & & \ddots & & y \\ 0 & & & & & & 1 \end{bmatrix}$$

The image of $[\ \ ]_m$ normalizes U_m. Let R_m denote the corresponding semidirect product. It is then an algebraic subgroup of N_m isomorphic to $\mathbf{G}_a \ltimes \mathbf{G}_a^{(m-1)}$. We will write the elements of R_m as $([y]_m; (x_1, \ldots, x_{m-1}))$ with y, x_i in $\mathbf{G}_a$. Let R_2 denote $\mathbf{G}_a \times \mathbf{G}_a$.

Let α and β denote respectively the cycles around 0 and 1 in $\mathbf{P}^1_{\mathbf{C}} - \{0, 1, \infty\}$, oriented in the usual way. Then $\pi_1 = \pi_1(\mathbf{P}^1_{\mathbf{C}} - \{0, 1, \infty\})$ is a free group on α and β. Now for each $k \geqslant 1$, define a representation $\lambda_k: \pi_1 \to R_{k+1}(\mathbf{C}) \subset \mathrm{GL}_{k+1}(\mathbf{C})$ by

$$\alpha \mapsto ([1]_{k+1}; (0, \ldots, 0)) \quad \text{and} \quad \beta \mapsto ([0]_{k+1}; (1, 0, \ldots, 0)).$$

Put $\Gamma_{k+1} = \lambda_k(\pi_1)$. It is a discrete, k-step nilpotent subgroup of $R_{k+1}(\mathbf{C})$. Note that Γ_3 is the $\mathbf{Z}$-points of the 3-dimensional Heisenberg group and that $\Gamma_2 = \mathbf{Z}^2 = \pi_1^{ab} = \Gamma_{k+1}^{ab}$, for every $k \geqslant 1$.

Let M_{k+1} denote the complex nilmanifold $\Gamma_{k+1} \backslash R_{k+1}(\mathbf{C})$.

THEOREM. (a) *The (multivalued map)* $\rho_k: \mathbf{P}^1_{\mathbf{C}} - \{0, 1, \infty\} \to R_{k+1}(\mathbf{C}) \subset N_{k+1}$, *given by*

$$x \mapsto ([L(x)]_{k+1}; \ln_1(x), \ln_2(x), \ldots, \ln_k(x)),$$

becomes well defined modulo Γ_{k+1}.

(b) *Each* M_{k+1} *comes equipped with a* $\mathbf{C}^*$*-action with the quotient being identified with* M_k. *If* p_k *denotes the corresponding projection* $M_{k+1} \to M_k$, *then we have the commutative diagram* (*for* $k \geqslant 2$)

$$\begin{array}{ccc} & \overset{\rho_k}{\nearrow} & M_{k+1} \\ \mathbf{P}^1_{\mathbf{C}} - \{0, 1, \infty\} & & \downarrow p_k \\ & \underset{\rho_{k-1}}{\rightarrow} & M_k \end{array}$$

(c) *There exists a holomorphic connection* ∇_k *on* $M_{k+1} \overset{p_k}{\to} M_k$ *such that* ρ_k *yields a flat section to the corresponding pullback (via* ρ_{k-1}*) bundle with connection on* $\mathbf{P}^1_{\mathbf{C}} - \{0, 1, \infty\}$.

PROOF. (a) The solution space to $(*)_k$ is spanned by $\{\ln_k, L^j/j! \mid 0 \leqslant j \leqslant k-1\}$. It is easy to check that the monodromy around 1 amounts to sending $\ln_k$ to

$\ln_k + L^{(k-1)}/(k-1)!$ and fixing the $(L^j/j!)$'s. Consequently, β acts on

$$v_k \overset{\text{def}}{=} \begin{pmatrix} \ln_k \\ L^{(k-1)}/(k-1)! \\ \vdots \\ L \\ 1 \end{pmatrix}$$

by multiplication on the left by $\rho_k(\beta)$.

The monodromy around 0 fixes $\ln_k$ and sends each $L^j/j!$ to

$$\sum_{0 \leq i \leq j} \frac{1}{(j-i)!} \frac{L^i}{i!}.$$

Thus α acts on v_k via $\rho_k(\alpha)$ on the left. The assertion (a) now follows, since λ_k sends x to the matrix whose $(l+1)$st column is

$$\begin{pmatrix} v_l(x) \\ 1 \\ 0 \end{pmatrix} \begin{matrix} \}l \\ \\ \}k-l. \end{matrix}$$

(b) Define $p_k \colon R_{k+1} \to R_k$ when $k > 2$ (resp. $k = 2$) by

$$([y]_{k+1}; (x_1, \ldots, x_k)) \to ([y]_k; (x_1, \ldots, x_{k-1}))$$

(resp. $([y]_3; (x_1, x_2)) \to (y, x_1)$).

In either case let i_k denote the injection $\mathbf{G}_a \to R_{k+1}$ given by $z \to ([0]_{k+1}; (0, \ldots, 0, z))$. Then we have the exact sequence

$$0 \to \mathbf{G}_a \xrightarrow{i_k} R_{k+1} \xrightarrow{p_k} R_k \to 1.$$

Note that since Γ_{k+1} is a discrete, k-step nilpotent subgroup of $R_{k+1}(\mathbf{C})$, the intersection of $i_k(\mathbf{Q})$ with Γ_{k+1} is a rank-1 free abelian group, and is generated by $i_k(t_k)$ for some t_k in $\mathbf{Q}$.

The map p_k makes sense on Γ_{k+1} and $p_k(\Gamma_{k+1}) = \Gamma_k$. Consequently we have the commutative exact diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathbf{C} & \xrightarrow{i_k} & R_{k+1}(\mathbf{C}) & \xrightarrow{p_k} & R_k(\mathbf{C}) & \to & 1 \\ & & \cup & & \cup & & \cup & & \\ 0 & \to & t_k\mathbf{Z} & \xrightarrow{i_k} & \Gamma_{k+1} & \xrightarrow{p_k} & \Gamma_k & \to & 1. \end{array}$$

Identifying $\mathbf{C}/t_k\mathbf{Z}$ with $\mathbf{C}^*$ via $z \mapsto \exp(2\pi i z/t_k)$, we get an action of $\mathbf{C}^*$ on each $M_{k+1} = \Gamma_{k+1} \backslash R_{k+1}(\mathbf{C})$. The quotient clearly identifies, via p_k, to $M_k = \Gamma_k \backslash R_k(\mathbf{C})$. Finally, unwinding the definition of $\{\rho_k\}$ we see that $\rho_{k-1} = p_k \circ \rho_k$.

(c) Let G_{k+1} (resp. C_{k+1}) denote the complex Lie group $i_k(t_k\mathbf{Z}) \backslash R_{k+1}(\mathbf{C})$ (resp. $i_k(t_k\mathbf{Z}) \backslash i_k(\mathbf{C})$), and let $\tilde{\Gamma}_{k+1}$ denote the image of Γ_{k+1} under the canonical projection $R_{k+1}(\mathbf{C}) \to G_{k+1}$. Then M_{k+1} is none other than the quotient of G_{k+1} by the left action of $\tilde{\Gamma}_{k+1}$. If Δ_{k+1} denotes the group $\tilde{\Gamma}_{k+1} \cdot C_{k+1}$ in G_{k+1}, then there exists a unique character μ_{k+1} of Δ_{k+1} which is trivial on $\tilde{\Gamma}_{k+1}$ and is the identity on C_{k+1}. The space $\mathfrak{L}_k$ of holomorphic sections of the line bundle associated to p_k: $M_{k+1} \to M_k$ can now be identified (as a right G_{k+1}-module) with the representation of G_{k+1} induced holomorphically by μ_{k+1}. This gives rise to an action π_k of Lie G_{k+1} on $\mathfrak{L}_k$. To define a connection we have to give a way to differentiate the sections in $\mathfrak{L}_k$ by the derivations on the base M_k. To do this we first note that Lie G_{k+1} can be realized, when $k > 2$ (resp. $k = 2$), as

$$\left\{ [y; x_1, x_2, \ldots, x_k] = \begin{pmatrix} 0 & x_1 & x_2 & \cdots & x_k \\ & 0 & y & \cdots & y \\ & & 0 & \ddots & \vdots \\ & & & \ddots & y \\ 0 & & & & 0 \end{pmatrix} \middle| \, x_1, \ldots, x_k, y \in \mathbf{C} \right\}$$

(resp. $\{(y, x) \mid y, x \in \mathbf{C}\}$). We see that the map Lie $G_{k+1} \to$ Lie G_k (coming from the differential of p_k) admits a vector space- (but not a Lie algebra-) section s_k given, when $k > 2$ (resp. $k = 2$), by $[y; x_1, \ldots, x_{k-1}] \mapsto [y; x_1, \ldots, x_{k-1}, 0]$ (resp. $(y, x) \mapsto [y; x, 0]$). Now define ∇_k to be the holomorphic connection defined by the action on $\mathfrak{L}_k$ via $\pi_k \circ s_k$ of the right invariant derivations on G_k.

It is a simple exercise to verify that, locally on $\mathbf{P}^1_{\mathbf{C}} - \{0, 1, \infty\}$, a section φ to the pullback bundle with connection via ρ_{k-1} is flat if and only if

$$\varphi \cdot \ln_{k-1}(x)\, dL(x) + d\varphi = 0.$$

Certainly, ρ_k gives rise to such a (flat) section. Q.E.D.

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