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A NOTE ON THE IRREDUCIBILITY OF LEBESGUE MEASURE WITH APPLICATIONS TO RANDOM WALKS ON THE UNIT CIRCLE

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ABSTRACT. Let μ be a probability measure on R. We say that a σ-finite measure λ is irreducible with respect to μ if there does not exist a Borel set A with $\mu(A)$, $\mu(A^c) > 0$ such that $\int_A \mu(A^c - x)\,\lambda(dx) = 0$. It is well known that the Lebesgue measure $m(dx)$ is irreducible with respect to any discrete measure whose support is R. We prove that every absolutely continuous measure is irreducible with respect to any probability measure whose support is R and give an application of this fact to random walks on the unit circle.

A well-known property of Lebesgue measure on the real line R is the following: Let A be a Borel set of positive Lebesgue measure and $\{\lambda_i\}_{i=1}^{\infty}$ a countable dense subset of R. Then $m((\bigcup_{i=1}^{\infty}(A - \lambda_i))^c) = 0$. (Here, m denotes the Lebesgue measure). In other words, if we try to cover the real line by translating a set of positive Lebesgue measure through a countable dense set, then we will miss at most a set of Lebesgue measure 0. We can see this fact from a different point of view. Let μ be a Borel probability measure concentrated on the countable dense set $\{\lambda_i\}_{i=1}^{\infty}$ and a Borel set with $m(A) > 0$. Then

$$\mu(A - x) > 0 \quad \text{a.e. } x\text{-}m(dx). \tag{1}$$

(Throughout this paper "a.e. x-$m(dx)$" will mean almost every x with respect to the measure m.) To see this, let x be a point such that $\mu(A - x) = 0$. Then $\lambda_i \notin A - x$ for every $i = 1, 2, \ldots$, i.e., $x \notin A - \lambda_i$ for every $i = 1, 2, \ldots$. Therefore, $x \notin \bigcup_{i=1}^{\infty}(A - \lambda_i)$. Since $m((\bigcup_{i=1}^{\infty}(A - \lambda_i))^c) = 0$, we have $\mu(A - x) > 0$ a.e. x-$m(dx)$. We say that a σ-finite measure λ is irreducible with respect to μ if there does not exist a set A with $\lambda(A)$, $\lambda(A^c) > 0$ such that $\int_A \mu(A^c - x)\,\lambda(dx) = 0$. Then (1) implies that any absolutely continuous measure is irreducible with respect to μ if μ is discrete and has support R. It seems obvious that (1) should hold for any probability measure μ whose support is R, not only for those discrete ones. But if one tries to prove this seemingly obvious fact using the same technique as we used when μ is discrete, the problem which will be encountered is that there is no such a fixed countable dense set $\{\lambda_i\}_{i=1}^{\infty}$ to use as when μ is discrete. In Theorem 1 we give a proof of this fact making essential use of Fubini's theorem. As an application of this, we show in Theorem 3 that a random walk on the unit circle with normalized

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Lebesgue measure as the initial distribution is ergodic if and only if the transition function does not have a lattice distribution.

THEOREM 1. *Let μ be a Borel probability measure on R with the property that $\mu(E) > 0$ for every open set $E \subseteq R$. Then every absolutely continuous measure is irreducible with respect to μ.*

PROOF. It is easy to see that $\mu(A - x)$ is a measurable function in x for every Borel set A. Therefore if we let $B = \{x\colon \mu(A - x) = 0\}$, then B is measurable. Consider,

$$0 = \int_B \mu(A - x)\, m(dx) = \iint \chi_B(x)\chi_{A-x}(y)\,\mu(dy)\, m(dx)$$

$$= \iint \chi_B(x)\chi_{A-y}(x)\, m(dx)\,\mu(dy),$$

so $\int \chi_B(x)\chi_{A-y}(x)\, m(dx) = 0$ for a.e. y-$\mu(dy)$. That is, $\int \chi_{B\cap(A-y)}(x)\, m(dx) = 0$ for a.e. y-$\mu(dy)$, i.e.,

$$m(B \cap (A - y)) = 0 \quad \text{a.e. } y\text{-}\mu(dy). \tag{2}$$

Since the support of $\mu = R$, we can choose at least a countable dense set $\{y_i\}_{i=1}^{\infty}$ such that (2) holds. Thus $m(B \cap (\bigcup_{i=1}^{\infty}(A - y_i))) = 0$. But $m((\bigcup_{i=1}^{\infty}(A - y_i))^c) = 0$, so $m(B) = 0$. Now, let λ be an absolutely continuous measure and A a Borel set with $\lambda(A)$, $\lambda(A^c) > 0$. Then $\mu(A - x) > 0$ a.e. x-$m(dx)$ in A^c. So $\mu(A - x) > 0$ a.e. x-$\lambda(dx)$ in A^c. This implies $\int_{A^c} \mu(A - x)\,\lambda(dx) > 0$.

REMARK. Theorem 1 can easily be generalized to the unit circle.

Let $X_0, X_1, \ldots$ be a Markov process on the unit circle S with initial distribution $\frac{1}{2\pi}m(dx)$ and transition function $p(x, dy)$. We say that $X_0, X_1, \ldots$ is a random walk if $p(x, dy) = p(dy - x)$. We say that $p(x, dy)$ has a lattice distribution if there exist $x_1, \ldots, x_n$ such that each x_i is a rational multiple of 2π and $p(0, \{x_i\}_{i=1}^n) = 1$. For an arbitrary random walk, let $\mu = \sum_{n=1}^{\infty} p^n(0, dy)/2^n$ where $p^n(x, dy)$ is the nth transition function. The following lemma is easy and the proof will be omitted.

LEMMA 2. *Let $p(x, dy) = p(dy - x)$ be a transition function and μ be defined as above. Then the support of μ is S if and only if $p(x, dy)$ does not have a lattice distribution.*

A theorem regarding the ergodicity of random walks reads as follows: a random walk on the unit circle with initial distribution $\frac{1}{2\pi}m(dx)$ is ergodic if and only if there does not exist a set A with $m(A)$, $m(A^c) > 0$ such that $\int_A p(x, A^c)\, m(dx) = \int_{A^c} p(x, A)\, m(dx) = 0$ (cf. [1, p. 143]). We are now ready to state

THEOREM 3. *A random walk on the unit circle with initial distribution $\frac{1}{2\pi}m(dx)$ and transition function $p(x, dy)$ is ergodic if and only if $p(x, dy)$ does not have a lattice distribution.*

PROOF. Suppose $p(x, dy)$ does not have a lattice distribution. If the random walk were not ergodic, then there exists a set A with $m(A)$, $m(A^c) > 0$ such that

$\int_A p(A^c - x)\, m(dx) = 0$. This implies that $\int_A p^n(A^c - x)\, m(dx) = 0$ for $n = 1, 2, \ldots$. Therefore,

$$\int_A \sum_{n=1}^{\infty} \frac{1}{2^n} p^n(A^c - x)\, m(dx) = 0,$$

i.e., $\int_A \mu(A^c - x)\, m(dx) = 0$. But by Theorem 1 and Lemma 2,

$$\int_A \mu(A^c - x)\, m(dx) > 0,$$

so the random walk is ergodic. The converse is trivial and we omit the proof.

References

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