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SOLUTION OF THE γ-SPACE PROBLEM

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ABSTRACT. This paper disproves the classic conjecture that every γ-space is quasi-metrizable.

A *quasi-metric* on a set X is a generalized metric d: $X \times X \to [0, \infty)$ satisfying the axioms $d(x, y) = 0 \Leftrightarrow x = y$ and $d(x, z) \leqslant d(x, y) + d(y, z)$, but not necessarily the axiom of symmetry [**N**, **W**]. As with a metric, the family of all sets $B_d(x, r) = \{y: d(x, y) < r\}$, for $r > 0$, form a neighbourhood base at each $x \in X$ for a topology on X. A space X with such a topology is called *quasi-metrizable*.

Following [**J1**], a (an *open*) *neighbournet* V on a space X is a binary relation on X such that, for each $x \in X$, the set $V[x]$ is a (an open) neighbourhood of x. A neighbournet V is called a *normal neighbournet* if there exists a sequence $\langle V_k: k \in \mathbf{N}\rangle$ of neighbournets with $V_{k+1}^2 \subseteq V_k$ for each $k \in \mathbf{N}$ and with $V_1 \subseteq V$. Clearly, if X is a quasi-metrizable space with quasi-metric d, and if for some $n \in \mathbf{N}$ and each $x \in X$ we have $B_d(x, 1/n) \subseteq V[x]$, then V is a normal neighbournet; for we may define the V_k by $V_k[x] = B_d(x, 2^{-k}/n)$.

With the above terminology, a T_1 space X is called a γ-*space* if there exists a decreasing sequence $\langle V_k: k \in \mathbf{N}\rangle$ of neighbournets (called a γ-*sequence*) such that, for each $x \in X$, the family $\{V_k^2[x]: k \in \mathbf{N}\}$ is a neighbourhood base at x [**LF**, **J1**]. Clearly every quasi-metrizable space is a γ-space since we may define the V_k by $V_k[x] = B_d(x, 2^{-k})$.

The question as to whether every γ-space is quasi-metrizable has been raised frequently (and often independently) in the literature, for example [**NČ**, **S**, **LN**, **LF**, **G**, **J2**], and is listed as Classic Problem VIII in [**TP**]. Indeed, a proof of this question was first claimed in the literature in 1943 [**R1**, p. 35]; however the argument given was incomplete and the author had later to strengthen his required conditions [**R2**]. Currently several partial solutions have been obtained [**G–K2**; **J2**; **B–K1**; **F2**].

In this paper we construct a Hausdorff counterexample to the γ-space conjecture (currently we know of no regular counterexample). If U^n is a normal neighbournet whenever U is a neighbournet on X, then the space X is said to be n-pretransitive [**FL**]. Our starting point will be the existence for each $n \in \mathbf{N}$, as proven in [**F1**], of a Hausdorff quasi-metrizable space X_n which is not n-pretransitive (such a space is the $(n + 1)$th power M^{n+1} of the Michael line M). Let U_n be an open neighbournet on

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X_n such that U_n^n is not a normal neighbournet. If we let $X = \bigcup_{n=1}^\infty X_n$ be the disjoint topological sum, and define the open neighbournet U on X by $U = \bigcup_{n=1}^\infty U_n$, then for no $k \in \mathbf{N}$ is U^k a normal neighbournet. On the other hand each X_n, and hence X, are quasi-metrizable spaces and thus γ-spaces. From the space X and open neighbournet U we construct the non-quasi-metrizable γ-space $\tilde{X}$ as follows.

The points of $\tilde{X}$ are the points of $X \cup X^2 \cup X^3 \cup \cdots \cup X^\omega$. The basic open sets of $\tilde{X}$ are all sets in X^n of the form $\{\langle x_1,\ldots,x_{n-1}\rangle\} \times A$ where A is open in X, for each $n \in \mathbf{N}$, together with all sets

$$\tilde{U}_{x;k} = \{x\} \cup \bigcup_{n=k}^{\infty} \left(\{\langle x_1,\ldots,x_{n-1}\rangle\} \times U^{n-k+1}[x_n]\right)$$

where $k \in \mathbf{N}$ and $x = \langle x_1, x_2, \ldots\rangle \in X^\omega$.

To show that $\tilde{X}$ is a γ-space, let $\langle V_k \colon k \in \mathbf{N}\rangle$ be a γ-sequence for X with $V_1 \subseteq U$ (for example, let $\langle W_k \colon k \in \mathbf{N}\rangle$ be any γ-sequence for X and take $V_k = W_k \cap U$). Define neighbournets $\tilde{V}_k$ on $\tilde{X}$ as follows: If $x = \langle x_1,\ldots,x_n\rangle \in X^n$ then $\tilde{V}_k[x] = \{\langle x_1,\ldots,x_{n-1}\rangle\} \times V_k[x_n]$, while if $x \in X^\omega$ then $\tilde{V}_k[x] = \tilde{U}_{x;k}$. Then $\langle \tilde{V}_k \colon k \in \mathbf{N}\rangle$ is a γ-sequence for $\tilde{X}$. For if $x = \langle x_1,\ldots,x_n\rangle \in X^n$ then clearly the sets $(\tilde{V}_k)^2[x] = \{\langle x_1,\ldots,x_{n-1}\rangle\} \times V_k^2[x_n]$, for $k \in \mathbf{N}$, form a neighbourhood base at x. Alternatively, if $x = \langle x_1, x_2,\ldots\rangle \in X^\omega$ then the sets

$$\begin{aligned}(\tilde{V}_k)^2[x] &= \{x\} \cup \bigcup_{n=k}^{\infty} \left(\{\langle x_1,\ldots,x_{n-1}\rangle\} \times V_k \circ U^{n-k+1}[x_n]\right) \\ &\subseteq \{x\} \cup \bigcup_{n=k}^{\infty} \left(\{\langle x_1,\ldots,x_{n-1}\rangle\} \times U^{n-k+2}[x_n]\right) \subseteq \tilde{V}_{k-1}[x],\end{aligned}$$

for $k \in \mathbf{N}$, form a neighbourhood base at x.

To show that $\tilde{X}$ is not quasi-metrizable, suppose d is a quasi-metric for $\tilde{X}$ and define x_n, $z_n \in X$ by induction on $n \in \mathbf{N}$ as follows.

Assume inductively that $x_1,\ldots,x_{n-1}$ have been defined. Since $\{\langle x_1,\ldots,x_{n-1}\rangle\} \times X \subseteq X^n \subseteq \tilde{X}$ is canonically homeomorphic to X, we may choose x_n, $z_n \in X$ such that $d(\langle x_1,\ldots,x_{n-1}, x_n\rangle, \langle x_1,\ldots,x_{n-1}, z_n\rangle) < 1/n$ but $z_n \notin U^n[x_n]$; for otherwise U^n would be a normal neighbournet.

Having completed the induction, let $x = \langle x_1, x_2,\ldots\rangle \in X^\omega$. Find $m \in \mathbf{N}$ such that $B_d(x, 2/m) \subseteq \tilde{U}_{x;1}$. Choose $n \geqslant m$ such that $d(x, \langle x_1,\ldots,x_n\rangle) < 1/m$; then $d(\langle x_1,\ldots,x_n\rangle, \langle x_1,\ldots,x_{n-1}, z_n\rangle) < 1/n \leqslant 1/m$ but $d(x, \langle x_1,\ldots,x_{n-1}, z_n\rangle) \geqslant 2/m$ since $\langle x_1,\ldots,x_{n-1}, z_n\rangle \notin \tilde{U}_{x;1}$. Thus d does not satisfy the triangle inequality. This proves that the space $\tilde{X}$ is not quasi-metrizable.

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