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THE ONLY GENUS ZERO n-MANIFOLD IS S^n

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ABSTRACT. All n-manifolds of regular genus zero, i.e. admitting a crystallization which regularly imbeds into S^2, are proved to be homeomorphic to S^n. A conjecture implying the Poincaré Conjecture in dimension four is also formulated.

SUNTO. Si dimostra che tutte le n-varietà di genere regolare zero, cioè aventi una cristallizzazione che si immerge regolarmente in S^2, sono omeomorfe a S^n. Si formula anche una congettura che implica quella di Poincaré in dimensione quattro.

1. Throughout this paper, we work in the PL category, for which we refer to **[RS]**; for graph theory, we refer to **[Har]**. $\cong$ denotes PL-homeomorphism.

An h-coloured graph (Γ, γ) is a multigraph Γ, regular of degree h, together with a coloration γ of the edges by h colours. If $\mathcal{K}$ is the colour set, and $\mathcal{B} \subset \mathcal{K}$, $\Gamma_{\mathcal{B}}$ will denote the subgraph of Γ generated by the edges e such that $\gamma(e) \in \mathcal{B}$. Given a colour $c \in \mathcal{K}$, $\hat{c}$ will denote the set $\mathcal{K} - \{c\}$. An h-coloured graph (Γ, γ) is said to be *contracted* if $\Gamma_{\hat{c}}$ is connected for each $c \in \mathcal{K}$.

To every $(n+1)$-coloured graph (Γ, γ), there corresponds an n-dimensional pseudocomplex $K(\Gamma)$, whose i-simplexes are in one-one correspondence with the connected components of the subgraphs $\Gamma_{\mathcal{B}}$ for all colour subsets $\mathcal{B}$ of cardinality $\#\mathcal{B} = n - i$. Note that, if (Γ, γ) is contracted, then $K(\Gamma)$ has exactly $n+1$ vertices. For every closed, connected n-manifold M, there exists at least one contracted $(n+1)$-coloured graph (Γ, γ) such that $|K(\Gamma)| \cong M$; such a graph is called a *crystallization* of M, and $K(\Gamma)$ a *contracted triangulation* of M. For the existence and equivalence theorems for crystallizations, see **[P, F, FG$_1$]**; these and other results are also summarized in **[FGG]**.

We recall the notion of regular genus of a manifold, defined in **[G$_3$]**, which generalizes the genus of a surface and Heegaard genus of a 3-manifold. A 2-cell imbedding **[Wh**, p. 40**]** $\iota: |\Gamma| \to F$ of an $(n+1)$-coloured graph (Γ, γ) into a closed surface F is said to be *regular* if there exists a cyclic permutation $\varepsilon = (\varepsilon_0, \ldots, \varepsilon_n)$ of the colour set, such that each region of ι is bounded by the image of a cycle, whose edges are alternatively coloured by ε_i, ε_{i+1} (i being an integer mod $n+1$). The *regular genus* $\rho(\Gamma)$ of (Γ, γ) is defined to be the least genus of a surface into which

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(Γ, γ) regularly imbeds. Given a closed n-manifold M, its *regular genus* (or simply *genus*) $\mathcal{G}(M)$ is defined as the integer

$$\mathcal{G}(M) = \min\{\rho(\Gamma) \mid (\Gamma, \gamma) \text{ is a crystallization of } M\}.$$

As usual, we shall identify a graph with its imbedded image.

[G$_3$, Corollary 7] asserts, among other things, that a 4-manifold of genus zero is simply-connected. We shall extend this result to dimension n. This permits us to compute $\mathcal{G}(\mathbf{S}^1 \times \mathbf{S}^n)$, and further to prove the following fact, which confirms the geometrical significance of this invariant.

THEOREM 1. *Let M be a closed, connected n-manifold; then*

$$\mathcal{G}(M) = 0 \Leftrightarrow M \cong \mathbf{S}^n.$$

REMARK 1. In view of Theorem 1, it would be interesting to study the behaviour of $\mathcal{G}$ with respect to connected sums. $\mathcal{G}$ is easily proved to be subadditive by direct construction. It is trivially additive in dimension 2; in dimension 3, the Heegaard genus—hence also the regular genus—is known to be additive too [**Hak**, §7]. If the same property held in dimension 4, as we conjecture, this would imply an affirmative answer to the 4-dimensional Poincaré Conjecture. In fact, as it is well known [**M**, §1.1; **Wa**; **C**], if M is a 4-dimensional homotopy sphere then, for a suitable nonnegative integer k, $M \# k(\mathbf{S}^2 \times \mathbf{S}^2) \cong \mathbf{S}^4 \# k(\mathbf{S}^2 \times \mathbf{S}^2)$. But this would imply that $\mathcal{G}(M) = 0$, whence $M \cong \mathbf{S}^4$.

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2. From now on, $\Delta_n = \{i \in \mathbf{Z} \mid 0 \leq i \leq n\}$ will be assumed as a colour set. For each $\mathfrak{B} \subset \Delta_n$, $g_{\mathfrak{B}}$ will denote the number of connected components of $\Gamma_{\mathfrak{B}}$.

LEMMA 1. *Let (Γ, γ) be a contracted $(n + 1)$-coloured graph, such that $\rho(\Gamma) = 0$, and $\varepsilon = (\varepsilon_0, \ldots, \varepsilon_n)$, a cyclic permutation of Δ_n associated to a regular imbedding ι of (Γ, γ) into $\mathbf{S}^2$. Let $\mathfrak{B} \subset \Delta_n$ contain at least three colours $\varepsilon_{i-1}, \varepsilon_i, \varepsilon_{i+1}$ consecutive in ε (i taken in $\mathbf{Z}_{n+1}$). Then $g_{\mathfrak{B}} = g_{\mathfrak{B}-\{\varepsilon_i\}}$.*

PROOF. As (Γ, γ) is contracted, $\Gamma_{\hat{\varepsilon}_i}$ is connected. Call γ' and ι' the restrictions of γ and ι respectively to the latter graph; then $(\Gamma_{\hat{\varepsilon}_i}, \gamma')$ is an n-coloured graph, regularly imbedded by ι' into $\mathbf{S}^2$. Namely, ι' is a 2-cell imbedding [**Wh**, Theorem 6.11], and colours $\varepsilon_{i-1}, \varepsilon_{i+1}$ are now contiguous in the corresponding permutation of $\Delta_n - \{\varepsilon_i\}$; hence, $(\varepsilon_{i-1}, \varepsilon_{i+1})$-coloured cycles bound regions of ι'.

Therefore, each edge coloured by ε_i joins two vertices of the same component of $\Gamma_{\{\varepsilon_{i-1},\varepsilon_{i+1}\}}$, thus also of the same component of $\Gamma_{\mathfrak{B}-\{\varepsilon_i\}}$. $\square$

LEMMA 2. *Let (Γ, γ) and ε be as in Lemma 1. Let further $\mathfrak{B} = \Delta_n - \mathfrak{B}'$, where $\mathfrak{B}'$ contains no two colours consecutive in ε. Then $g_{\mathfrak{B}} = 1$.*

PROOF. Follows from Lemma 1, by induction on $\#\mathfrak{B}'$. $\square$

PROPOSITION 1. *For a closed, connected n-manifold M, $\mathcal{G}(M) = 0 \Rightarrow M$ is simply-connected.*

PROOF. Obvious for $n = 2$. For $n > 2$, if (Γ, γ) of Lemma 2 is a crystallization of M, and $\mathfrak{B} = \Delta_n - \{i, j\}$ with i and j not consecutive in ε, then there is only one component of $\Gamma_{\mathfrak{B}}$. Then [$\mathbf{G_2}$, §6, Proposition 9] proves the statement. □

As conjectured in [$\mathbf{FG_2}$, §6], we have

COROLLARY 1. $\mathcal{G}(\mathbf{S}^1 \times \mathbf{S}^n) = 1$.

PROOF. $\mathcal{G}(\mathbf{S}^1 \times \mathbf{S}^n) > 0$ by Proposition 1.

In order to see that $\mathcal{G}(\mathbf{S}^1 \times \mathbf{S}^n) \leq 1$, consider the following construction of a crystallization of $\mathbf{S}^1 \times \mathbf{S}^n$, which generalizes [$\mathbf{G_2}$, Figures 1, 8] [$\mathbf{FG_2}$, Figures 4, 7] and is obtained by applying the method illustrated in [$\mathbf{FG_2}$, §2].

Take $2n + 4$ vertices v_j^i ($i \in \Delta_1$, $j \in \Delta_{n+1}$). Join v_j^i with v_{j+1}^i ($i \in \Delta_1$, $j \in \Delta_{n+1}$) by an edge coloured by j. Put a further edge coloured by $n + 1$ between v_0^i and v_{n+1}^i ($i \in \Delta_1$) if n is even, between v_0^0 and v_{n+1}^1 and between v_0^1 and v_{n+1}^0 if n is odd. Finally, join v_j^0 with v_j^1 ($j \in \Delta_{n+1}$) by n edges coloured by the n colours not yet used around those vertices.

The fact that such a graph can be regularly imbedded into the torus—with respect to every cyclic permutation of Δ_{n+1}—follows from the equality $\mathfrak{g}_{\{i,j\}} = n$ for all $i, j \in \Delta_{n+1}$, $i \neq j$ (see [**FGG**, §5]). □

3. Proof of Theorem 1. It is trivial to see that $M \cong \mathbf{S}^n \Rightarrow \mathcal{G}(M) = 0$, as $\mathbf{S}^n$ admits a standard crystallization consisting of two vertices joined by $n + 1$ differently coloured edges; this graph obviously imbeds regularly into $\mathbf{S}^2$ with respect to every cyclic permutation of Δ_n.

The proof of the converse implication consists of some general considerations followed by three parts, relative to the cases (A) n odd, (B) n even and $\neq 4$, (C) $n = 4$.

In the following construction, which was first introduced in [$\mathbf{G_1}$], M is an arbitrary closed n-manifold (not necessarily of genus zero), (Γ, γ) a given crystallization of it, and K the relative contracted triangulation.

In the vertex set $V = \{v_0, \ldots, v_n\}$ of K, assume that v_i corresponds to $\Gamma_{\hat{i}}$. For each nonvoid subset W of V, set $W' = V - W$, and call K_W the contracted subcomplex of K generated by W. If $W = h + 1$, then $\dim K_W = h$. Furthermore, if $\mathfrak{B}$ is the subset of Δ_n such that $W = \{v_i \mid i \in \mathfrak{B}\}$ and $\mathfrak{B}' = \Delta_n - \mathfrak{B}$, then the number of h-simplexes of K_W equals $\mathfrak{g}_{\mathfrak{B}'}$; this is easy to check. Now let L be the largest subcomplex of $\operatorname{Sd} K$, disjoint from $\operatorname{Sd} K_W \cup \operatorname{Sd} K_{W'}$.[2] Then L, whose space is a closed $(n - 1)$-manifold, splits K into two complementary subcomplexes, N_W and $N_{W'}$ say, having L as common boundary. Moreover, $|N_W|$ and $|N_{W'}|$ are regular neighbourhoods, in $|K|$, of $|K_W|$ and $|K_{W'}|$ respectively. Observe that, in dimension three, if $\sharp W = 2$, then $(|N_W|, |N_{W'}|)$ is a Heegaard splitting of M.

From now on, the hypothesis $\rho(\Gamma) = 0$ will be assumed, and $\iota\colon |\Gamma| \to \mathbf{S}^2$ will denote a regular imbedding of (Γ, γ); w.l.o.g., ι can be assumed to be associated to the fundamental cyclic permutation $\varepsilon = (0, 1, \ldots, n)$.

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[2] Sd means "barycentric subdivision of"; it carries every pseudocomplex to a simplicial complex.

(A) $n = 2r + 1, r \geqslant 0$.

Set $\mathfrak{B} = \{2k+1 \mid 0 \leqslant k \leqslant r\}$, $\mathfrak{B}' = \Delta_n - \mathfrak{B}$; call W, W' the corresponding subsets of V. By Lemma 2, $\mathfrak{g}_{\mathfrak{B}'} = \mathfrak{g}_{\mathfrak{B}} = 1$, whence K_W and $K_{W'}$ consist of exactly one r-simplex each. Therefore $|N_W|$ and $|N_{W'}|$ are closed $(2r+1)$-balls; they cover M, and meet in their common boundary $|L|$. Thus $M \cong \mathbf{S}^{2r+1}$.

(B) $n = 2r, r \neq 2$.

$\mathfrak{B}, \mathfrak{B}', W, W'$ as in case (A). Here, Lemma 2 only assures that $\mathfrak{g}_{\mathfrak{B}'} = 1$, hence that $|N_W|$ is a $2r$-ball. The $2r$-complex N_W, whose boundary L has a $(2r-1)$-sphere as space, has the homotopy type of the $(r-1)$-complex $K_{W'}$. These facts, applied to the Mayer-Vietoris homology sequence of $K = K_W \cup K_{W'}$ and $L = K_W \cap K_{W'}$, together with Poincaré duality, imply that $M \cong |K|$ is a homology sphere. Therefore, as a consequence of Proposition 1 and of the Hurewicz isomorphism theorem, M is even a homotopy sphere. This, which holds for all r, implies that $M \cong \mathbf{S}^{2r}$ when $r \neq 2$, by the generalized Poincaré Conjecture (Smale, Stallings and Zeeman).

(C) $n = 4$.

$\mathfrak{B} = \{1,3\}$, $\mathfrak{B}' = \{0,2,4\}$; W, W' as before. Again, $\mathfrak{g}_{\mathfrak{B}'} = 1$ implies that $|N_W|$ is a 4-ball.

In order to show that $|N_{W'}|$ is a 4-ball too, let us examine $K_{W'}$ in some detail. Since $\mathfrak{g}_{\{1,3,4\}} = \mathfrak{g}_{\{0,1,3\}} = 1$ by Lemma 2, $K_{\{v_0,v_2\}}$ and $K_{\{v_2,v_4\}}$ are formed by one 1-simplex each. Hence all triangles forming $K_{W'}$ have two edges in common; then $K_{W'}$ will be a cone over the 1-pseudocomplex $K_{\{v_0,v_4\}}$ if it consists of as many triangles as there are edges in $K_{\{v_0,v_4\}}$. But this is actually the case, as $\mathfrak{g}_{\{1,2,3\}} = \mathfrak{g}_{\{1,3\}}$ by Lemma 1. Therefore $|K_{W'}|$ is collapsible, $|N_{W'}|$ is a 4-ball (by Whitehead's theorem [**RS**, Corollary 3.27]), and $M \cong \mathbf{S}^4$. $\square$

For $n \geqslant 2$ we have

Corollary 2_n. *Let (Γ, γ) be a contracted $(n+1)$-coloured graph such that $\rho(\Gamma_{\hat{\imath}}) = 0$ for each $i \in \Delta_n$. Then $|K(\Gamma)|$ is a manifold.*

Proof. For each $i \in \Delta_n$, $\Gamma_{\hat{\imath}}$ is connected and of regular genus zero. If $n = 2$, $\Gamma_{\hat{\imath}}$ is a cycle and hence represents $\mathbf{S}^1$. If $n \geqslant 3$, the fact that $|K(\Gamma_{\hat{\imath}})| \cong \mathbf{S}^{n-1}$ is assured by Corollary 3_{n-1}. This proves that, for each vertex v of $K(\Gamma)$, $|lk(v, \mathrm{Sd}\, K(\Gamma))| \cong \mathbf{S}^{n-1}$, and this suffices to prove the statement (compare [**F**, Proposition 16]). $\square$

Corollary 3_n. *Let (Γ, γ) be a connected $(n+1)$-coloured graph such that $\rho(\Gamma) = 0$. Then $|K(\Gamma)| \cong \mathbf{S}^n$.*

Proof. By eliminating a suitable number of dipoles of type 1 [**FG$_1$**, §3] one obtains a contracted graph (Γ', γ'). Now let $\iota: |\Gamma| \to \mathbf{S}^2$ be a regular imbedding of (Γ, γ) into $\mathbf{S}^2$ relative to the cyclic permutation ε. Then by [**FG$_2$**, Lemma 1] there exists also an imbedding $\iota': |\Gamma'| \to \mathbf{S}^2$ relative to the same ε.

If $|K(\Gamma')|$ is a manifold, i.e. if (Γ', γ') is a crystallization, then $|K(\Gamma')| \cong |K(\Gamma)|$. But $|K(\Gamma')|$ is actually a manifold by Corollary 2_n, since ι' induces a regular imbedding of each $(\Gamma_{\hat{\imath}}, \gamma|_{\Gamma_{\hat{\imath}}})$ into $\mathbf{S}^2$. Therefore $|K(\Gamma)| \cong |K(\Gamma')| = \mathbf{S}^n$ by Theorem 1 applied to (Γ', γ'). $\square$

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