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HOMEOMORPHISM GROUPS OF SOME DIRECT LIMIT SPACES

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ABSTRACT. Let F be either of the spaces $R^\infty = \varinjlim R^n$ or $Q^\infty = \varinjlim Q^n$ where R denotes the reals and Q the Hilbert cube. Let $\mathcal{H}(M)$ be the homeomorphism group of a connected F-manifold M with the compact-open topology. We prove that $\mathcal{H}(M)$ is separable, Lindelöf, paracompact, non-first-countable, and not a k-space.

Let R denote the reals and Q the Hilbert cube. Let F denote either of $R^\infty = \varinjlim R^n$ or $Q^\infty = \varinjlim Q^n$. It is known that F is paracompact [**7**, III.1] but not first countable [**6**, p. 391], hence not metrizable. By an F-manifold we mean a paracompact space which is locally homeomorphic to F.

Work by Heisey and Liem shows that the behavior of F-manifolds is similar to that of metrizable, infinite dimensional manifolds. Let l_2 denote separable, infinite dimensional Hilbert space. Just as for l_2-manifolds, F-manifolds are triangulable, stable on multiplication by the model, and classified by homotopy type; also, each connected F-manifold embeds as an open subset of F (see Heisey [**8** and **9**]). Recently, α-approximation theorems [**10, 12**] and unknotting theorems [**11, 12**] have been achieved for F-manifolds, similar to those for Q-manifolds.

For any space X, let $\mathcal{H}(X)$ denote the space of all homeomorphisms of X onto itself with the compact-open topology. $\mathcal{H}(X)$ is a group under composition of functions, though not necessarily a topological group. In the metrizable case, Renz [**13**] has shown that $\mathcal{H}(l_2)$ and $\mathcal{H}(Q)$ are contractible. (Recall that $l_2 \cong \prod_1^\infty R$ [**1**].) Further, a theorem completed by Ferry [**3**] states that the homeomorphism group of a compact Q-manifold is an l_2-manifold.

The problem we address here is the nature of the homeomorphism group of an F-manifold M. In [**6**] Heisey proved that $\mathcal{H}(F)$ is contractible. Previous work [**4**] by the author shows that $\mathcal{H}(M)$ is F-stable; that is, $\mathcal{H}(M) \times F \cong \mathcal{H}(M)$. As noted above, stability is a property shared by F-manifolds. In this paper we show that, also like M, $\mathcal{H}(M)$ is separable, Lindelöf, paracompact, and non-first-countable. However, we also show that $\mathcal{H}(M)$ is not a k-space, and hence not an F-manifold.

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Call a space X *weakly second countable* if there is a countable collection $\mathcal{D} = \{D_k | k = 1, 2, \dots\}$ of subsets of X, not necessarily open, such that for each $x \in X$ and each open set U of X containing x there is an integer k such that $x \in D_k \subset U$.

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Clearly each second countable space is weakly second countable, as well as any countable union of second countable spaces. Thus, F is weakly second countable, but not second countable.

PROPOSITION 1. *A weakly second countable space is separable and Lindelöf.*

The proof of Proposition 1 follows the proofs for second countability [2, VIII, 6.3, 7.3].

A space X is a *countable direct limit of compact metric spaces* (CDLCMS) if $X = \varinjlim C_n$, where each C_n is a compact metric subspace of C_{n+1}. Examples are R^n, F, and any connected F-manifold [7, III.2]. Since a CDLCMS is Hausdorff and regular [5, 4.1, 4.3] we have

PROPOSITION 2 [5, 1.1]. *If X is a CDLCMS, then $\mathcal{H}(X)$ is Hausdorff and regular.*

NOTATION. The identity map on a space X will be denoted by id_X. If $A, B \subset X$, then $(A, B) = \{h \in \mathcal{H}(X) | h(A) \subset B\}$. The closure of A in X is written $\mathrm{cl}_X(A)$.

THEOREM 1. *If X is a CDLCMS, then $\mathcal{H}(X)$ is weakly second countable.*

THEOREM 2. *If X is a CDLCMS, then $\mathcal{H}(X)$ is separable, Lindelöf, and paracompact.*

REMARK. Theorems 1 and 2 apply to spaces X which are connected F-manifolds, as noted above.

Theorem 2 follows easily from Theorem 1 by applying Propositions 1 and 2 and [2, VIII, 6.5].

PROOF OF THEOREM 1. Write $X = \varinjlim C_n$ as in the definition of CDLCMS. Choose a countable collection $\mathcal{D} = \{D_k | k = 1, 2, \dots\}$ of subsets of X satisfying

(i) Each D_k is a compact neighborhood in some C_n;

(ii) $\mathcal{D}$ contains a (compact) basis for each C_n.

Let $f \in (K, W) \subset \mathcal{H}(X)$, where K is compact and W is open. (Thus (K, W) is a typical subbasic open set in $\mathcal{H}(X)$.) By [5, 2.4] there are integers m, n such that $K \subset C_n$ and $f(C_n) \subset C_m$. Now, $K \subset f^{-1}(W) \cap C_n$, which is open in C_n, so there is a finite set M of integers such that

$$K \subset \bigcup_{k \in M} D_k \subset f^{-1}(W) \cap C_n.$$

By similar reasoning, there is a finite set N of integers such that

$$f\Big(\bigcup_{k \in M} D_k\Big) \subset \bigcup_{l \in N} D_l \subset W \cap C_m.$$

Thus, $f \in (\bigcup_{k \in M} D_k, \bigcup_{l \in N} D_l) \subset (K, W)$.

Now, let $\mathcal{U} \subset \mathcal{H}(X)$ be an arbitrary open set containing f. Then there is some basic open set $\mathcal{O} = \bigcap_{i=1}^{j} (K_i, W_i)$ such that $f \in \mathcal{O} \subset \mathcal{U}$. As above, for each i, there are finite sets M_i and N_i of integers such that

$$f \in \bigcap_{i=1}^{j} \Big(\bigcup_{k \in M_i} D_k, \bigcup_{l \in N_i} D_l\Big) \subset \mathcal{O} \subset \mathcal{U}.$$

But the collection of all possible sets of this form is countable, and the theorem is proved.

We now turn to two further theorems. As before, M denotes an F-manifold (not necessarily connected).

THEOREM 3. *$\mathcal{H}(M)$ is not first countable.*

THEOREM 4. *$\mathcal{H}(M)$ is not a k-space.*

Note that R^∞ is a factor of M [9, Theorem 1, 8, Theorem 7]. We prove Theorems 3 and 4 for the special case $M = R^\infty$ (Theorems 5 and 6 below). The general case then follows from the following

PROPOSITION 3. *For Hausdorff spaces X and Y, the map $\phi\colon \mathcal{H}(X) \to \mathcal{H}(X \times Y)$ defined by $\phi(h) = h \times \mathrm{id}_Y$ is a closed embedding.*

The proof of Proposition 3 is routine and will be omitted.

THEOREM 5. *$\mathcal{H}(R^\infty)$ is not first countable.*

PROOF. Let $\mathcal{C} = \{(K_i, W_i) | i = 1, 2, \dots\}$ be any countable collection of subbasic neighborhoods of id_{R^∞} in $\mathcal{H}(R^\infty)$. It will suffice to show that there is a neighborhood U of $0 = (0, 0, \dots) \in R^\infty$ such that no finite intersection $\bigcap_{i=1}^n (K_i, W_i)$ is contained in $(\{0\}, U)$.

Since R^∞ is separable and locally path-connected, the collection $\mathcal{P}$ of all path components of all sets of the form

$$R^\infty - \left(\bigcup_{k=1}^{r} K_{i(k)}\right); \quad \text{or}$$

$$\bigcap_{j=1}^{m} W_{i(j)}; \quad \text{or}$$

$$\left(\bigcap_{j=1}^{m} W_{i(j)}\right) - \left(\bigcup_{k=1}^{r} K_{i(k)}\right),$$

where m and r take on all integer values, is countable. Since R^∞ is not first countable, $\mathcal{P}$ does not form a basis for R^∞ at 0, and hence there is a neighborhood U of 0 such that if $0 \in P$ and $P \in \mathcal{P}$, then $P \not\subset U$.

Let n be given. We must show that $\bigcap_{i=1}^n (K_i, W_i) \not\subset (\{0\}, U)$. The argument breaks down into three cases, all handled similarly.

Case (i): $0 \notin \bigcup_{i=1}^n K_i$. Set $V = R^\infty - (\bigcup_{i=1}^n K_i)$. Let P be the path component of V containing 0. So $P \in \mathcal{P}$ and there is a point $y \in P - U$. The map taking 0 to y is an embedding into R^∞ homotopic to inclusion. Using Lemma 3.1 of [12] we extend to a homeomorphism h of R^∞, fixed outside P, and taking 0 to y. Then $h \in [\bigcap_{i=1}^n (K_i, W_i)] - (\{0\}, U)$.

Case (ii): $0 \in \bigcap_{i=1}^n K_i$. Set $V = \bigcap_{i=1}^n W_i$ and repeat the arguments of Case (i).

Case (iii): $0 \in (\bigcap_{i=1}^m K_i) - (\bigcup_{i=m+1}^n K_i)$, $1 \le m < n$. Set $V = (\bigcap_{i=1}^m W_i) - (\bigcup_{i=m+1}^n K_i)$ and repeat the preceding argument.

Thus the proof is complete.

THEOREM 6. *$\mathcal{H}(R^\infty)$ is not a k-space.*

PROOF. Consider R^∞ as a topological vector space with basis $\mathcal{B} = \{e_n | n = 1, 2, \dots\}$, where $e_n = (0, \dots, 0, 1, 0, \dots)$, 1 in the nth component. Let $\mathcal{L}(R^\infty)$

denote the space of all linear maps of R^∞ into itself, with the compact-open topology. Define $\mathcal{LH}(R^\infty) = \mathcal{L}(R^\infty) \cap \mathcal{H}(R^\infty)$.

It is easy to see that the map $\psi\colon \mathcal{L}(R^\infty) \to \prod_1^\infty R^\infty$ defined by $\psi(f) = (f(e_1), f(e_2), \ldots)$ is a homeomorphism taking $\mathcal{LH}(R^\infty)$ onto the subset

$$Y = \{(x_1, x_2, \ldots) | \{x_1, x_2, \ldots\} \subset R^\infty \text{ is a basis}\}.$$

Now, $\mathcal{LH}(R^\infty)$ is a closed subspace of $\mathcal{H}(R^\infty)$, so in order to prove Theorem 6 it suffices to prove that $\mathcal{LH}(R^\infty)$, or Y, is not a k-space.

We construct a set A in Y whose intersection with every compact set is closed, but which is not itself closed. This construction is similar to [6, II-1(b)].

Let $e = (e_1, e_2, \ldots) \in Y$. For positive integers r and j, define

$$x_r^j = \frac{1}{r} \cdot e_j = \left(0, \ldots, 0, \frac{1}{r}, 0, 0, \ldots\right) \in R^\infty.$$

For $j \geq 2$, define $A_j \subset Y$ by

$$A_j = \left\{(e_1 + x_r^j, e_2, \ldots, e_{j-1}, e_j + e_k, e_{j+1}, \ldots) \in \prod_1^\infty R^\infty | k \geq 2, r \leq k\right\}.$$

We show that indeed $A_j \subset Y$. Let $j, k \geq 2$ and $r \leq k$ be given. Of the elements

$$e_1 + x_r^j, e_2, \ldots, e_{j-1}, e_j + e_k, e_{j+1}, \ldots,$$

for $m \notin \{1, j, k\}$, e_m is the only one whose mth component is nonzero. Thus it suffices to show that the remaining elements form a linearly independent set, and that all the elements span R^∞.

Case (i): $j \neq k$. Assume

$$\begin{aligned} 0 &= \lambda_1(e_1 + x_r^j) + \lambda_2(e_j + e_k) + \lambda_3 e_k \\ &= (\lambda_1, 0, \ldots, 0, \lambda_2 + \lambda_1/r, 0, \ldots, 0, \lambda_2 + \lambda_3, 0, \ldots). \end{aligned}$$

Then $\lambda_i = 0$ for $1 \leq i \leq 3$, and the set is linearly independent.

Let $x = (x_1, x_2, \ldots) = \sum_{i=1}^\infty x_i e_i \in R^\infty$. It is routine to verify that

$$x = x_1(e_1 + x_r^j) + \left(x_j - \frac{x_1}{r}\right)(e_j + e_k) + \left(x_k - x_j + \frac{x_1}{r}\right)e_k + \sum_{i \neq 1, j, k} x_i e_i.$$

Case (ii): $j = k$. If $0 = \lambda_1(e_1 + x_r^j) + \lambda_2(2e_j) = (\lambda_1, 0, \ldots, 0, \lambda_1/r + 2\lambda_2, 0, \ldots)$, then $\lambda_1 = 0 = \lambda_2$, so the set is linearly independent.

Again, one can easily check that

$$\sum_{i=1}^\infty x_i e_i = x_1(e_1 + x_r^j) + \frac{1}{2}\left(x_j - \frac{x_1}{r}\right)(2e_j) + \sum_{i \neq 1, j} x_i e_i.$$

Thus each $A_j \subset Y$ and we define $A = \bigcup_{j=2}^\infty A_j \subset Y$. This is the set we desired.

If $C \subset Y$ is compact, then $C \subset \prod_{i=1}^\infty R^{n_i}$, for some integers n_i [5, 2.4]. We show that $A \cap (\prod_1^\infty R^{n_i})$ is empty or finite. Thus $A \cap C$ will be empty or finite, and hence closed in C.

For a given integer j, if A_j meets $\prod_1^\infty R^{n_i}$, then $e_1 + x_r^j \in R^{n_1}$, for some r. Hence $j \leq n_1$, and only finitely many of the sets A_j meet $\prod_1^\infty R^{n_i}$. Fix j between

2 and n_1 and suppose

$$(e_1 + x_r^j, e_2, \ldots, e_{j-1}, e_j + e_k, e_{j+1}, \ldots) \in A_j \cap \left(\prod_1^\infty R^{n_i}\right).$$

Then $e_j + e_k \in R^{n_j}$, so $k \leq n_j$, and there are only finitely many choices for k. By definition of A_j, $r \leq k$, so there are only finitely many choices for r. That is, $A_j \cap (\prod_1^\infty R^{n_i})$ is finite.

It remains to show that A is not closed in Y. We show that $e \in \mathrm{cl}_Y(A) - A$. Clearly $e \notin A$, since x_r^j is never 0. Choose a basic neighborhood of e in Y. We may assume this neighborhood is of the form $V = [\prod_{i=1}^n (e_i + U) \times \prod_{n+1}^\infty R^\infty] \cap Y$, where $U = (\prod_{l=1}^\infty U_l) \cap R^\infty$ and U_l is a neighborhood of 0 in R [6, II-1(a)]. Choose $r \geq 2$ such that $1/r \in U_{n+1}$. Then the point

$$(e_1 + x_r^{n+1}, e_2, \ldots, e_n, e_{n+1} + e_r, e_{n+2}, \ldots)$$

is in $A_{n+1} \cap V$. So $e \in \mathrm{cl}_Y(A)$.

Thus Theorem 6 is proved.

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