

GRAPHS WITH SUBCONSTITUENTS CONTAINING $L_3(p)$

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ABSTRACT. Let Γ be a finite connected undirected graph, G a vertex-transitive subgroup of $\text{aut}(\Gamma)$, $\{x, y\}$ an edge of Γ and $G_i(x, y)$ the subgroup of G fixing every vertex at a distance of at most i from x or y . We show that if the stabilizer G_x contains a normal subgroup inducing $L_3(p)$, p a prime, on the set of vertices adjacent to x , then $G_5(x, y) = 1$.

Let Γ be an undirected graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$ and let G be a subgroup of $\text{aut}(\Gamma)$. For each $x \in V(\Gamma)$, we denote by $\Gamma(x)$ the set of vertices adjacent to x , by $G(x)$ the stabilizer of x in G and, for each $i \in \mathbb{N}$, by $G_i(x)$ the subgroup $\{a \in G(x) \mid a \in G(u) \text{ for all } u \in V(\Gamma) \text{ with } \partial(x, u) \leq i\}$ where $\partial(x, u)$ denotes the distance between x and u . An s -path (for $s \in \mathbb{N}$) is an $(s+1)$ -tuple (x_0, \dots, x_s) of vertices such that $x_i \in \Gamma(x_{i-1})$ for $1 \leq i \leq s$ and $x_i \neq x_{i-2}$ for $2 \leq i \leq s$. Let $G(x_0, \dots, x_s) = G(x_0) \cap \dots \cap G(x_s)$ and $G_i(x_0, \dots, x_s) = G_i(x_0) \cap \dots \cap G_i(x_s)$ for each s -path (x_0, \dots, x_s) and each $i \in \mathbb{N}$.

If H is a group acting on a set X , a an element of H , we denote by H^X the permutation group induced by H on X and by a^X the permutation of X induced by a . In the above context, the permutation group $G(x)^{\Gamma(x)}$ (for $x \in V(\Gamma)$) is known as the subconstituent of G at x .

We prove the following

THEOREM. *Let p be an arbitrary prime. Let Γ be a finite undirected connected graph, $\{x, y\}$ an edge of Γ and G a subgroup of $\text{aut}(\Gamma)$ acting transitively on $V(\Gamma)$ such that $G(x)^{\Gamma(x)} \supseteq L_3(p)$ (where $L_3(p)$ is to be understood as acting on the $1 + p + p^2$ points of the associated projective plane). Let s be the largest integer such that G acts transitively on the set of all s -paths in Γ . Then $2 \leq s \leq 3$, $G_3(x, y) = 1$ if $s = 2$ and $G_5(x, y) = 1$ if $s = 3$.*

Let Γ be an arbitrary finite undirected connected graph, $\{x, y\}$ an arbitrary edge of Γ and G a subgroup of $\text{aut}(\Gamma)$ acting transitively on $V(\Gamma)$ such that $G(x)^{\Gamma(x)}$ is primitive. In [6, 7], Tutte showed that $|G(x)|$ is bounded (in fact, that $|G(x)|$ divides 48) if $|\Gamma(x)| = 3$. Our Theorem may be seen as a step in the efforts to generalize this result to graphs of arbitrary valency. By [2, (2.3)], $|G_1(x, y)|$ is in general a power of some prime. From this fact it follows easily that $G_1(x, y) = 1$ or that $G(x, y)^{\Gamma(x)}$ has a nontrivial normal subgroup of prime power order. This directs

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attention (in particular) to the case that $G(x)^{\Gamma(x)} \cong L_n(q)$. The case $n = 2$ was solved (i.e. a bound found for $|G(x)|$ depending only on q) in [3, 4]; see also [10]. The case $n \geq 3$ has proved more stubborn. A partial result was proved in [9]. See p. 214 of that paper for a discussion of examples associated with the Chevalley groups of type A_{2n-2} and F_4 (with $n = 3$). Other examples associated with the Chevalley groups of type A_n and D_n are described below. Our Theorem may also be of some relevance to the "pushing-up" problem; see, for instance, [1]. In [5], Goldschmidt solved the related problem of determining the structure of $G(x)$ and $G(y)$ in the case that Γ is a trivalent graph and $G^{E(\Gamma)}$ (but not necessarily $G^{\Gamma(\Gamma)}$) is transitive. Some of his ideas play a role in the proof of our Theorem.

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We begin the proof. Assume that Γ and G fulfill the hypotheses. For each $x \in V(\Gamma)$, $G(x)$ induces a projective plane on $\Gamma(x)$. For $y \in \Gamma(x)$, we denote by $[x : y]$ the set of those lines in this plane passing through y . If z is a second vertex in $\Gamma(x)$, then $[x : y, z]$ will denote the (unique) line of $[x : y]$ containing z .

Let (w, x, y) be an arbitrary 2-path. If $G_1(w, x) \leq G_1(y)$, then $G_1(w, x) = G_1(x, y)$ and hence $G_1(w, x) \leq \langle G(w, x), G(x, y) \rangle = G(x)$. Since G contains elements exchanging w and x , we have $G_1(w, x) \leq G(w)$ too. But since Γ is connected, $\langle G(w), G(x) \rangle$ acts transitively on $E(\Gamma)$ and so $G_1(w, x) = 1$. Thus we may assume that $G_1(w, x) \not\leq G_1(y)$. Since $G_1(w, x) \leq G_1(x) \leq G(x, y)$, the order of $G_1(w, x)$ is divisible by p . By [2, (2.3)], we have

LEMMA 1. *For each edge $\{x, y\}$, $G_1(x, y)$ is a p -group and $O_p(G(x, y)^{\Gamma(y)}) = O_p(G(x, y))^{\Gamma(y)}$. \square*

Suppose that $G_1(x)$ acts intransitively on $\Gamma(y) - \{x\}$ (which is certainly the case when $s = 2$). Since $G_1(x)^{\Gamma(y)} \leq G(x, y)^{\Gamma(y)}$, it follows that $G_1(x)^{[y:x]} = 1$. Let ϕ denote the homomorphism from $G(x, y)$ to $G(x, y)^{[y:x]}$ defined by $\phi(a) = a^{[y:x]}$. Let K denote the kernel of ϕ . The group K is normal in $G(x, y)$, so if $K^{[x:y]} \neq 1$, then $K^{[x:y]}$ is transitive and, in particular, $p + 1 \mid |K^{[x:y]}|$. Since $|K^{\Gamma(y)}| \mid (p - 1)p^2$ and $G_1(y)^{[x:y]} = 1$, we conclude that in fact $K^{[x:y]} = 1$. Thus ϕ induces an isomorphism from $G(x, y)^{[x:y]}$ to $G(x, y)^{[y:x]}$ which, because $G(x, y)^{[x:y]} \cong L_2(p)$, is in turn induced by a bijection from $[x : y]$ to $[y : x]$ which we denote by $\phi_{(x,y)}$. In particular, $a(z) \in [y : x, z]$ for each $a \in G(w, x, y)$ and each $z \in \phi_{(x,y)}([x : w, y])$ which implies that $s = 2$. (Thus $\phi_{(x,y)}$ is only defined when $s = 2$. Note that if we replace $L_3(p)$ by $L_n(q)$, $n \geq 3$ arbitrary, in the statement of the Theorem, $G(x, y)$ induces on $[x : y]$ and $[y : x]$ a projective space of dimension $n - 2$. When $s = 2$ we still have a natural isomorphism from $G(x, y)^{[x:y]}$ to $G(x, y)^{[y:x]}$, but if $n \geq 4$, we can only conclude that it is induced by a collineation or a correlation between these two projective spaces. Both cases actually occur in interesting examples: If $G = D_n(q)$ and Γ is the graph whose vertices are the maximal subspaces of the associated polar space, two being joined by an edge if their intersection is maximal in both, then the isomorphism is induced by a collineation; if $G = A_n(q)$ with $n \geq 4$ and Γ is the graph whose vertices are the maximal and minimal subspaces (i.e. points and

copoints) of the associated projective space, two being joined by an edge if one contains the other, then the isomorphism is induced by a correlation.)

Suppose that $G_1(x)^{[y:x]} \neq 1$ (so $s \geq 3$). Since $G_1(x)^{\Gamma(y)} \trianglelefteq G(x, y)^{\Gamma(y)}$, we conclude that $G_1(x)^{[y:x]} \cong L_2(p)$ except perhaps when $p \leq 3$ in which case $|G_1(x)^{[y:x]}| = p + 1$ is also possible. Suppose that this actually holds. Let $a \in G(x, y)$ be an arbitrary p -element such that $a^{[x:y]} \neq 1$. Then $a^{[x:y]} \notin G_1(y)^{[x:y]}$ and, in particular, $a \notin G_1(y)$. Since $G_1(x)^{\Gamma(y)} \geq O_p(G(x, y)^{\Gamma(y)})$, there exists a p -element $b \in G_1(x)$ such that ab has fixed points in $\Gamma(y) - \{x\}$. Note that both $(ab)^{\Gamma(x)}$ and $(ab)^{\Gamma(y)}$ are p -elements; since $G_1(x, y)$ is a p -group, ab is a p -element too. Among the fixed points of ab in $\Gamma(y)$ there is exactly one, say z , such that $(ab)^{[y:z]} = 1$. Thus there exists an element $c \in G_1(z)$ such that $abc \in G_1(y)$. It follows that $(ab)^{\Gamma(z)} = (abc)^{\Gamma(z)} \in G_1(y)^{\Gamma(z)}$. Since ab is a p -element, $(ab)^{[z:y]} = 1$. Now choose an element $d \in G_1(y)$ acting nontrivially on $[x:y]$ and set $e = [d, ab]$. Then $e^{[x:y]}$ is a nontrivial p' -element but $e^p \in G_1(y, z)$ so $|e|$ is a power of p . With this contradiction, we conclude that $G_1(x)^{[y:x]} \cong L_2(p)$ even when $p \leq 3$.

Continuing to assume that $s \geq 3$, we let ϕ denote the homomorphism from $G_1(x)$ to $G_1(x)^{[y:x]}$ defined by $\phi(a) = a^{[y:x]}$. Let K be the kernel of ϕ . Since $K^{[w:x]} \trianglelefteq G_1(x)^{[w:x]} \cong L_2(p)$, $p + 1 \mid |K^{[w:x]}|$ if $K^{[w:x]} \neq 1$. But $|K/G_1(x, y)| \mid (p - 1)p^2$ and $G_1(x, y)^{[w:x]} = 1$ (since otherwise, because $G_1(x, y) \trianglelefteq G_1(x)$, $G_1(x, y)^{[w:x]}$ would be transitive which contradicts the fact that $G_1(x, y)$ is a p -group) so $K^{[w:x]} = 1$ too. Hence ϕ induces an isomorphism from $G_1(x)^{[w:x]}$ to $G_1(x)^{[y:x]}$ which, because $G_1(x)^{[w:x]} \cong L_2(p)$, is in turn induced by a bijection $\phi_{(w,x,y)}$ from $[w:x]$ to $[y:x]$. Choose $a \in G(w, x, y)$ and $v \in \Gamma(w) - \{x\}$ and let M be the subgroup of $G_1(x)$ fixing $[w:v, x]$. Then $\phi_{(w,x,y)}([w:v, x])$ is the unique line in $[y:x]$ fixed by M and $\phi_{(w,x,y)}(a([w:v, x]))$ the unique line in $[y:x]$ fixed by aMa^{-1} , i.e. $a(\phi_{(w,x,y)}([w:v, x]))$. Thus $\phi_{(w,x,y)}$ commutes with the action of $G(w, x, z)$. In particular, $G(v, w, x, y)$ fixes $\phi_{(w,x,y)}([w:v, x])$ and so (see [8]) $s = 3$. (Thus $\phi_{(w,x,y)}$ is only defined when $s = 3$. We point out that if $L_3(p)$ is replaced by $L_n(p)$, $n \geq 3$ arbitrary, in the statement of the Theorem, we still have an isomorphism from $G_1(x)^{[w:x]}$ to $G_1(x)^{[y:x]}$ which, however, can be shown by an easy argument (see [9]) to be induced by a collineation between the projective spaces induced on $[w:x]$ and $[y:x]$ so we do not have the problem mentioned above in the case $s = 2$.)

DEFINITION. Let (x_0, \dots, x_{s+1}) be an arbitrary $(s + 1)$ -path in Γ . We call (x_0, \dots, x_{s+1}) crooked if $\phi_{(x_1, \dots, x_s)}([x_1 : x_0, x_2]) \neq [x_s : x_{s-1}, x_{s+1}]$. If (x_0, \dots, x_r) is an arbitrary path of length $r \geq s + 1$, we call (x_0, \dots, x_r) crooked if (x_i, \dots, x_{i+s+1}) is crooked for $0 \leq i \leq r - s - 1$.

LEMMA 2. Let (x_0, \dots, x_{2s}) be an arbitrary crooked $2s$ -path. If $a \in G(x_1, \dots, x_{2s-1})$ is a p -element fixing $[x_1 : x_0, x_2]$ and $[x_{2s-1} : x_{2s-2}, x_{2s}]$, then $a \in G_1(x_s)$.

PROOF. The element a fixes $[x_s : x_{s-1}, x_{s+1}]$, $\phi_{(x_1, \dots, x_s)}([x_1 : x_0, x_2])$ and $\phi_{(x_{2s-1}, \dots, x_s)}([x_{2s-1} : x_{2s-2}, x_{2s}])$. Since (x_0, \dots, x_{2s}) is crooked, these lines do not contain a common point. Since a is a p -element, a must act trivially on $\Gamma(x_s)$. \square

LEMMA 3. Let (x_0, \dots, x_5) be a crooked 5-path. If $s = 2$, then there are elements in $O_p(G(x_1)) \cap G(x_3)$ not fixing $[x_3 : x_2, x_4]$. If $s = 3$, then $G_1(x_0, x_1) \cap G(x_3, x_4) \not\leq G_1(x_2)$ or $G_1(x_0, x_1, x_2) \leq G_1(x_3)$; in the former case, $G_1(x_0, x_1) \cap G(x_3, x_4)$ contains elements not fixing $[x_4 : x_3, x_5]$.

PROOF. Since $O_p(G(x_1))^{\Gamma(x_2)} = O_p(G(x_1, x_2))^{\Gamma(x_2)}$, $O_p(G(x_1)) \cap G(x_3)$ certainly contains elements acting nontrivially on $\Gamma(x_2)$. If $s = 2$, then such elements do not fix $[x_3 : x_2, x_4]$ by Lemma 2.

Let $s = 3$. The last claim follows from Lemma 2 so we need only show that $G_1(x_0, x_1) \cap G(x_3, x_4) \not\leq G_1(x_2)$ or $G_1(x_0, x_1, x_2) \leq G_1(x_3)$. Since $G_1(x_0, x_1) \leq G(x_0, x_1, x_2)$ and $G(x_0, x_1, x_2)$ acts transitively on $\Gamma(x_2) - \{x_1\}$, there certainly exist elements in $G_1(x_0, x_1) \cap G(x_3) - G_1(x_2)$. Let a be one which, we may assume, does not fix x_4 . Since $a \in G_1(x_1)$, a acts trivially on $[x_1 : x_2]$ and hence on $\phi_{(x_1, x_2, x_3)}([x_1 : x_2]) = [x_3 : x_2]$ too. But $G_1(x_1, x_2)$ acts trivially on $[x_3 : x_2]$ too. Thus if $G_1(x_0, x_1, x_2) \not\leq G(x_4)$, then $ab \in G_1(x_0, x_1) \cap G(x_3, x_4) - G_1(x_2)$ for some suitable element $b \in G_1(x_0, x_1, x_2)$. Hence we may suppose that $G_1(x_0, x_1, x_2) \leq G(x_4)$, i.e. that $G_1(x_0, x_1, x_2)$ acts trivially on $[x_3 : x_2, x_4]$. Since $G_1(x_2)^{[x_1 : x_2]} \cong L_2(p)$, there exists an element $c \in G(x_0) \cap G_1(x_2)$ inducing a permutation of order p on $[x_1 : x_2]$ and hence on $[x_3 : x_2]$ too. Since (x_0, \dots, x_4) is crooked, c does not fix $[x_3 : x_2, x_4]$. Since c normalizes $G_1(x_0, x_1, x_2)$, $G_1(x_0, x_1, x_2)$ acts trivially on at least two lines in $[x_3 : x_2]$. It follows that $G_1(x_0, x_1, x_2) \leq G_1(x_3)$. \square

Suppose that $ZO_p(G(x, y))$ (i.e. the center of $O_p(G(x, y))$) is not contained in $G_m(x, y)$ for some edge $\{x, y\}$ and some m . Then there exists an edge $\{u, v\}$ and an element $a \in ZO_p(G(x, y))$ such that $G_m(u, v) \leq O_p(G(x, y))$ and $a \in G(u) - G(v)$. We have $G_m(u, v) = G_m(u, v)^a = G_m(u, a(v))$ and so $G_m(u, v) \leq \langle G(u, v), G(u, a(v)) \rangle = G(u)$. Since there exist elements in G exchanging u and v , $G_m(u, v) \leq G(v)$ as well. Thus $G_m(u, v) = 1$. Let $m = 3$ if $s = 2$ and $m = 5$ if $s = 3$. We may thus suppose from now on that $ZO_p(G(x, y)) \leq G_m(x, y)$ for every edge $\{x, y\}$.

Now let (x_0, \dots, x_{s+1}) be an arbitrary crooked $(s + 1)$ -path. Since G acts transitively on the set of s -paths in Γ , there exists an element $g \in G$ such that $g(x_i) = x_{i+1}$ for $0 \leq i \leq s$. Let $x_i = g^i(x_0)$ for every $i \in \mathbb{Z}$. (Of course, since Γ is finite, there are only finitely many distinct x_i .) Let $W = (\dots, x_{-1}, x_0, x_1, x_2, \dots)$. Then W is a crooked path.

Note that by Lemma 2, $G_1(W)$ is the subgroup of $G(W)$ generated by all the p -elements of $G(W)$. We suppose for the time being that this subgroup is trivial. Thus we can find an element $a \in ZO_p(G(x_0, x_1))$ not contained in $ZO_p(G(x_1, x_2))$. Let $\alpha \in \mathbb{N}$ be maximal such that $a \in G(x_0, x_1, \dots, x_{\alpha+1})$. Since $ZO_p(G(x_0, x_1)) \leq G_m(x_0, x_1)$, $\alpha \geq m$ and $a^{g^i} \in O_p(G(x_0, x_1))$ for $|i| \leq m$. In particular, $[a, a^g] = 1 = [a, a^{g^2}]$.

Suppose $s = 2$. If $a(x_{\alpha+2}) \notin [x_{\alpha+1} : x_\alpha, x_{\alpha+2}]$, then $a \notin G_1(x_\alpha)$. But a fixes $[x_\alpha : x_{\alpha-1}, x_{\alpha+1}]$ and $\phi_{(x_{\alpha-1}, x_\alpha)}([x_{\alpha-1} : x_{\alpha-2}, x_\alpha])$. These lines are distinct since W is crooked and so a acts trivially on $[x_\alpha : x_{\alpha-1}]$. It follows that a acts trivially on

$[x_\alpha : x_{\alpha-1}, x_{\alpha+1}]$ but on no other line in $[x_\alpha : x_{\alpha-1}]$. Thus a^g fixes the vertices in $[x_{\alpha+1} : x_\alpha, x_{\alpha+2}]$ but has no other fixed points in $\Gamma(x_{\alpha+1})$. In particular, $[a, a^g] \notin G_1(x_{\alpha+1})$. This contradicts the observation above that $[a, a^g] = 1$. Hence $a(x_{\alpha+2}) \in [x_{\alpha+1} : x_\alpha, x_{\alpha+2}]$. If $s = 3$, this same conclusion follows analogously from the fact that $[a, a^{g^2}] = 1$. By Lemma 2, we have $a \in G_1(x_0, \dots, x_\alpha)$ if $s = 2$ and $a \in G_1(x_0, \dots, x_{\alpha-1})$ if $s = 3$. Finally, choose an element $b \in ZO_p(G(x_0, x_1))$ not contained in $ZO_p(G(x_{-1}, x_0))$ and let $\beta \in \mathbb{N}$ be maximal such that $b \in G(x_{-\beta}, x_{-\beta+1}, \dots, x_0, x_1)$. Just as for a , we have $b(x_{-\beta-1}) \in [x_{-\beta} : x_{-\beta-1}, x_{-\beta+1}]$ and $b \in G_1(x_{-\beta+1}, \dots, x_0, x_1)$ if $s = 2$, $b \in G_1(x_{-\beta+2}, \dots, x_0, x_1)$ if $s = 3$.

For each $i \in \mathbb{Z}$, let $a_i = a^{g^i}$ and $b_i = b^{g^i}$. If $[a, b_{\beta+1}] = 1$, then $a \in ZO_p(G(b_{\beta+1}(x_0), x_1))$ and so (since $b_{\beta+1}(x_0) \in [x_1 : x_0, x_2] - \{x_0\}$) $a \in ZO_p(G(x_1, x_2))$ which contradicts the choice of a . Thus $[a, b_{\beta+1}] \neq 1$. But $b_{\beta+1} \in ZO_p(G(x_{\beta+1}, x_{\beta+2}))$ and $a \in G_1(x_0, \dots, x_{\alpha-s+2})$ so $a \in O_p(G(x_{\beta+1}, x_{\beta+2}))$ and hence $[a, b_{\beta+1}] = 1$ after all unless $\alpha - s + 2 \leq \beta$. By considering the element $[a_{-\alpha-1}, b]$, we conclude analogously that $\beta - s + 2 \leq \alpha$. Hence $\alpha = \beta$ if $s = 2$ and $\beta - 1 \leq \alpha \leq \beta + 1$ if $s = 3$. Without loss of generality, we may assume that $\alpha = \beta$ or $\beta + 1$.

Let $c = [a, b_{\beta+1}]$. We have $c = a \cdot (a^{-1})^{b_{\beta+1}} \in ZO_p(G(x_1))$ (since $ZO_p(G(u, x_1)) \leq G_1(u, x_1) \leq O_p(G(x_1))$ for each $u \in \Gamma(x_1)$) as well as $c = b_{\beta+1}^a \cdot b_{\beta+1}^{-1} \in ZO_p(G(x_{\alpha+1}))$ (whether $\alpha = \beta$ or $\beta + 1$). Let C denote the centralizer in G of the element c . First let $s = 2$. By Lemma 3, there exists an element d in $O_p(G(x_1)) \cap G(x_3) - G_1(x_2)$. Since d and $a_{-\alpha+2} \in O_p(G(x_1)) \leq C$ and $b_{\beta+3} \in O_p(G(x_{\alpha+1})) \leq C$, we have $\langle d, a_{-\alpha+2}, b_{\beta+3} \rangle \leq C(x_3)$. Since the element d does not map $[x_3 : x_2, x_4]$ to itself, $C(x_3)^{\Gamma(x_3)} \cong L_3(p)$. In particular, $C(x_2, x_3)$ contains an element e which does not map $\phi_{(x_2, x_3)}([x_2 : x_1, x_3])$ to itself. It follows that $\langle e, a_{-\alpha+1}, b_{\beta+2} \rangle \leq C(x_2)$ acts transitively on $\Gamma(x_2)$. Since Γ is connected, C therefore acts transitively on $E(\Gamma)$. This contradicts the fact that $c \neq 1$.

We conclude that $s = 3$. If $G_1(x_0, x_1, x_2) \leq G_1(x_3)$, then g normalizes $G_1(x_0, x_1, x_2)$ and so $G_1(x_0, x_1, x_2) \leq G_1(W) = 1$ which contradicts the fact that $\alpha \in G_1(x_0, \dots, x_{\alpha-1})$. By Lemma 3, therefore, there exists an element $d \in G_1(x_0, x_1) \cap G(x_3, x_4)$ not mapping $[x_4 : x_3, x_5]$ to itself. Thus $\langle d, b_{\beta+4}, a_{-\alpha+3} \rangle \leq C(x_4)$ acts transitively on $\Gamma(x_4)$. Moreover, d^g and $a_{-\alpha+4} \in O_p(G(x_1)) \leq C$ and $b_{\beta+5} \in O_p(G(x_{\alpha+1})) \leq C$ (even when $\alpha = 5$ since in this case $\beta = 5$ too and $b \in G_1(x_{-\beta+1})$). Thus $C(x_5)$ acts transitively on $\Gamma(x_5)$ and so C acts transitively on $E(\Gamma)$. Again this contradicts the fact that $c \neq 1$. With this contradiction we conclude that $G_1(W) \neq 1$.

Let T_p be the functor which assigns to each group the subgroup generated by its p -elements. Choose t maximal such that $T_p(G(x_0, \dots, x_t)) \not\leq G(x_{t+1})$. (Since there are only finitely many distinct x_i , t certainly exists.) Then $T_p(G(x_0, \dots, x_{t+1})) \leq T_p(G(x_1, \dots, x_{t+2}))$ and so g normalizes $T_p(G(x_0, \dots, x_{t+1}))$. It follows that $G_1(x_1, \dots, x_t) \leq T_p(G(x_0, \dots, x_{t+1})) = T_p(G(W)) = G_1(W) \leq G_1(x_1, \dots, x_t)$. Let N denote the normalizer of $G_1(x_1, \dots, x_t)$ in G .

Let $s = 2$. Since $O_p(G(x_1)) \cap G(x_3) \not\leq G(x_4)$, we have $t \geq 3$. Suppose $T_p(G(x_0, \dots, x_t))$ contains an element a such that $a(x_{t+1}) \notin [x_t : x_{t-1}, x_{t+1}]$. Of

course, $a \notin G_1(x_{t-1})$. Since a fixes both $[x_{t-1} : x_{t-2}, x_t]$ and $\phi_{(x_{t-2}, x_{t-1})}([x_{t-2} : x_{t-3}, x_{t-1}])$, a acts trivially on $[x_{t-1} : x_{t-2}]$. Thus $x_{t+1} \neq a \cdot a^g \cdot a^{-1}(x_{t+1}) \in [x_t : x_{t-1}, x_{t+1}]$. Let $b_1 = a \cdot a^g \cdot a^{-1}$. By the choice of t (and the fact that $g \in N$), we have $T_p(G(x_0, \dots, x_t)) \not\leq G(x_{-1})$. It follows that $N(x_0, x_1)$ contains a p -element b_2 such that $x_{-1} \neq b_2(x_{-1}) \in [x_0 : x_{-1}, x_1]$. Thus $\langle a, b_1, b_2^{g^2} \rangle \leq N(x_t)$ acts transitively on $\Gamma(x_t)$. Since Γ is connected, $\langle g, N(x_t) \rangle$ thus acts transitively on $V(\Gamma)$ which contradicts the fact that $G_1(W) \neq 1$. We conclude that $a(x_{t+1}) \in [x_t : x_{t-1}, x_{t+1}]$ for each $a \in T_p(G(x_0, \dots, x_t))$.

Let $s = 3$. If $G_1(x_0, x_1, x_2) \leq G_1(x_3)$, then $G_1(x_0, x_1, x_2) = G_1(W)$ and so $G(x_0, x_1, x_2) \leq N$. But then $\langle G(x_0, x_1, x_2), G(x_0, x_1, x_2)^{g^2} \rangle \leq N(x_2)$ acts transitively on $\Gamma(x_2)$. Since $g \in N$, we conclude that $G_1(x_0, x_1, x_2) = 1$ which contradicts our conclusion that $G_1(W) \neq 1$. Thus we may assume by Lemma 3 that $t \geq 5$. We can argue now just as in the previous paragraph that $a(x_{t+1}) \in [x_t : x_{t-1}, x_{t+1}]$ for every element $a \in T_p(G(x_0, \dots, x_t))$.

In both cases (i.e. $s = 2$ and $s = 3$) we thus have that for each $i \in \mathbb{Z}$, $N(x_i)$ maps $[x_i : x_{i-1}, x_{i+1}]$ to itself and $N(x_i)^{[x_i : x_{i-1}, x_{i+1}]} \cong L_2(p)$. Let Δ be the graph with vertex set x_0^N and edge set $\{x_0, x_1\}^N$. Then $|\Delta(x_0)| = p + 1$ and N acts transitively on the set of all $(t + 1)$ -paths in Δ . By [2, (3.15)], we have $t \leq 6$.

Let $s = 3$. Since $G_1(x_1, \dots, x_t) = G_1(W)$, we have $G(x_1, \dots, x_t) \leq N$ and hence $G(x_{-1}, \dots, x_4) \leq G(x_{-1}, \dots, x_{t-2}) = G(x_1, \dots, x_t)^{g^{-2}} \leq N$. Since, as we have already seen, $G_1(x_0, x_1, x_2) \not\leq G_1(x_3)$, there exists by Lemma 3 an element in $G_1(x_0, x_1) \cap G(x_3, x_4) \leq G(x_{-1}, \dots, x_4) \leq N(x_4)$ which does not map $[x_4 : x_3, x_5]$ to itself. With this contradiction, we conclude that $s = 2$.

We claim that G acts transitively on the set of crooked $(t - 1)$ -paths in Γ . Let (u_0, \dots, u_{t-1}) and (v_0, \dots, v_{t-1}) be any two; we want to find an element in G mapping the one to the other. Since G acts transitively on the set of all 2-paths in Γ , we may assume that $u_i = v_i$ for $0 \leq i \leq 2$. By Lemma 3, $G_1(u_0) \cap G(u_2)$ acts transitively on $[u_2 : u_1] - \{\phi_{(u_1, u_2)}([u_1 : u_0, u_2])\}$. Since (u_0, \dots, u_3) and (v_0, \dots, v_3) are both crooked, we may thus assume that $v_3 \in [u_2 : u_1, u_3]$. But then $G_1(u_1)$ contains an element mapping u_3 to v_3 so we may just assume that $u_3 = v_3$. We may assume too that $t - 1 \geq 4$.

Again by Lemma 3, $G_1(u_1) \cap G(u_3)$ acts transitively on $[u_3 : u_2] - \{\phi_{(u_2, u_3)}([u_2 : u_1, u_3])\}$; thus we may assume that $v_4 \in [u_3 : u_2, u_4]$ and we can find an element in G mapping (x_0, \dots, x_3) to (u_0, \dots, u_3) and $[x_3 : x_2, x_4]$ to $[u_3 : u_2, u_4]$. Since $T_p(G(x_{3-t}, \dots, x_3))$ acts transitively on $[x_3 : x_2, x_4] - \{x_2\}$, there exists an element in $G(u_0, \dots, u_3)$ mapping u_4 to v_4 . Thus we may assume that $u_4 = v_4$ and that $t - 1 = 5$.

Once again by Lemma 3, there exists an element a in $G_1(u_2) \cap G(u_4) - G_1(u_3)$. Since G acts transitively on crooked 3-paths and $G_1(u_2, u_3) \not\leq G_1(u_1)$, $G_1(u_2, u_3)$ acts transitively on $[u_1 : u_0, u_2]$. Thus $G_1(u_2, u_3)$ contains an element b such that $ab \in G(u_0)$. By Lemma 2, $\langle ab \rangle$ acts transitively on $[u_4 : u_3] - \{\phi_{(u_3, u_4)}([u_3 : u_2, u_4])\}$; thus we may assume that $v_5 \in [u_4 : u_3, u_5]$ and we can find an element in G mapping (x_0, \dots, x_4) to (u_0, \dots, u_4) and $[x_4 : x_3, x_5]$ to $[u_4 : u_3, u_5]$. Since $T_p(G(x_{4-t}, \dots, x_4))$ acts transitively on $[x_4 : x_3, x_5] - \{x_3\}$, there exists an

element in $G(u_0, \dots, u_4)$ mapping u_5 to v_5 . Thus G does in fact act transitively on the set of all crooked $(t-1)$ -paths in Γ .

Note that the parameter t is defined for each crooked path $(\dots, x_{-1}, x_0, x_1, x_2, \dots)$ such that there exists an element $g \in G$ with $g(x_i) = x_{i+1}$ for each $i \in \mathbb{Z}$. We may assume that $(\dots, x_{-1}, x_0, x_1, x_2, \dots)$ is chosen among all such crooked paths so that the value of t is maximal; thus for each crooked t -path (u_0, u_1, \dots, u_t) such that there exists an element $h \in G$ with $h(u_i) = u_{i-1}$ for $1 \leq i \leq t$, h normalizes $G_1(u_1, \dots, u_t)$. Let $w \in \Gamma(x_1) - [x_1 : x_0, x_2] - \phi_{(x_2, x_1)}([x_2 : x_1, x_3])$ be arbitrary. Since $(w, x_1, x_2, \dots, x_t)$ is crooked and G acts transitively on the set of crooked $(t-1)$ -paths in Γ , there exists an element $h \in G$ such that $h(x_1, \dots, x_t) = (w, x_1, x_2, \dots, x_{t-1})$. The element h normalizes $G_1(x_1, \dots, x_t)$ and so $hg \in G(x_1)$ lies in N but does not map $[x_1 : x_0, x_2]$ to itself. This contradiction concludes the proof of the Theorem.

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