A RESULT ON THE SET-THEORETIC COMPLETE INTERSECTION PROBLEM

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ABSTRACT. This article establishes a certain arithmetical criterion for set-theoretic complete intersection for irreducible analytic space curves.

Introduction. The object of this article is to study the set-theoretic complete intersection problem for the analytic space curves over an algebraically closed field k of characteristic 0. Let τ be a map from the power series ring of three variables $k[[x, y, z]]$ to $k[[t]]$ with

$$\tau(x) = t^l + \dots, \quad \tau(y) = t^m + \dots, \quad \tau(z) = t^n + \dots$$

Let P be the kernel of the map τ . The problem is to investigate if $P = \sqrt{(f, g)}$ for some suitable f and g . Note that an automorphism in $k[[x, y, z]]$ will reduce $\tau(x), \tau(y)$ and $\tau(z)$ to polynomials in t . Thus an affirmative answer to the set-theoretic complete intersection for the algebraic space curves will imply an affirmative answer to the present problem, while a solution to the present problem will certainly throw light to the global problem of the algebraic space curves. Our result is Theorem 1 which gives some partial solution to the analytic problem.

THEOREM 1. *If $l < m$, $(l, m) = 1$ and $(l - 2)m < n$, then the ideal P is a set-theoretic complete intersection.*

PROOF. Note that the conductor (t^{c^*}) of $k[[t]]$ over $k[[\tau(x), \tau(y)]]$ is with $c^* = (l - 1)(m - 1) < (l - 1)m$. Thus if $(l - 1)m \leq n$ then P defines a plane curve. We may henceforth assume that $n \notin$ the semigroup generated by l and m and the following: $(l - 2)m < n < (l - 1)m$. We may choose t properly with

$$\tau(x) = t^l, \quad \tau(y) = t^m + \dots, \quad \tau(z) = t^n + \dots$$

We shall view $(\tau(y), \tau(z))$ as a point P_i in the affine plane over an algebraic closure of $k[[t]]$ (i.e., the Puiseux field). Let ω be an l th primitive root of unity. The following conjugations

$$t \rightarrow \omega^i t, \quad i = 1, \dots, l,$$

will produce l points $P_1, \dots, P_l$ in the affine plane. We shall construct a polynomial $f(y, z)$ of the following form over $k[[t]]$

$$f(y, z) = y^{l-1} + \alpha_1 z + \alpha_2 y^{l-2} + \dots + \alpha_{l-1} + \alpha_l$$

with $f(P_i) = 0$ for $i = 1, \dots, l$.

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First note that if for $0 \leq i < (l-1)$ we have $n \equiv im \pmod{l}$ then we have for some positive s , $n = im + sl$. Hence $n \in$ the semigroup generated by l and m . We must then have

$$n \equiv (l-1)m \pmod{l}.$$

Let $\alpha_1 = t^{(l-1)m-n}\alpha_1^*$ and $\alpha_i = t^{(i-1)m}\alpha_i^*$ for $i = 2, \dots, l$. The conditions $f(P_i) = 0$ for $i = 1, \dots, l$ are linear equations in the indeterminates α_i^* . Let us factor out the common factor $t^{(l-1)m}$ from these equations. Then the matrix of the leading coefficients is the following nonsingular matrix

$$\begin{bmatrix} (\omega^m)^{l-1}, & (\omega^m)^{l-2}, & \dots, & 1 \\ (\omega^{2m})^{l-1}, & (\omega^{2m})^{l-2}, & \dots, & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1, & 1, & \dots, & 1 \end{bmatrix}.$$

Thus there exist solutions for $\alpha_1^*, \dots, \alpha_l^*$ in $k[[t]]$. Similarly, we may construct a polynomial $g(y, z)$ of the following form over $k[[t]]$

$$g(y, z) = z + \beta_2 y^{l-2} + \dots + \beta_l$$

with $g(P_i) = 0$, $i = 2, \dots, l$.

Let $F(y, z) = \Pi$ (all conjugates of $f(y, z)$), $G(y, z) = \Pi$ (all conjugates of $g(y, z)$). Then clearly $F, G \in P \subset k[[x, y, z]]$. We shall prove that $\sqrt{(F, G)} = P$.

It follows from the forms of F, G that

$$F = y^{(l-1)l} \pmod{x}, \quad G = z^l \pmod{x}.$$

Thus we conclude that the variety V defined by (F, G) contains no plane curve in the yz plane defined by (x) . Let W be any irreducible curve in the variety V . The parameterization of W can be taken as follows with fractional exponents allowed

$$\tau^*(x) = t^l, \quad \tau^*(y) = r_2 t^{m^*} + \dots, \quad \tau^*(z) = r_3 t^{n^*} + \dots$$

Then $(\tau^*(y), \tau^*(z))$ can be considered as a point in the affine plane over the algebraic closure of $k[[t]]$. In other words $(\tau^*(y), \tau^*(z)) \in \mathcal{V}$ (a conjugate of f) $\cap \mathcal{V}$ (a conjugate of g). However, it is easy to see these two curves intersect at only $(l-1)$ points of $\{P_1, \dots, P_l\}$. Thus $(\tau^*(y), \tau^*(z))$ is a conjugate of $(\tau(y), \tau(z))$. Clearly we have $\sqrt{(F, G)} = p$. Q.E.D.

EXAMPLE. If $l = 3$, then P is a set-theoretic complete intersection.

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