CERTAIN SCHUR-HADAMARD MULTIPLIERS IN THE SPACES C_p

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ABSTRACT. Let f be a continuously differentiable function on $[-1, 1]$ satisfying $|f'(t)| \leq C|t|^\alpha$ for some $0 < C, \alpha < \infty$ and all $-1 \leq t \leq 1$, and let $\lambda = (\lambda_i) \in l_r$ satisfy $-1 \leq \lambda_i \leq 1$ for all i . Then

$$a_{f,\lambda} = \left(\frac{f(\lambda_i) - f(\lambda_j)}{\lambda_i - \lambda_j} \right)$$

is a Schur-Hadamard multiplier from C_{p_1} into C_{p_2} for all p_1, p_2 satisfying $1 \leq p_2 \leq 2 \leq p_1 \leq \infty$ and $p_2^{-1} \leq p_1^{-1} + \alpha/r$.

1. Introduction . Let C_∞ be the space of all compact operators on l_2 with the operator norm. For $1 \leq p < \infty$ let C_p be the Banach space of all $x \in C_\infty$ for which $\|x\|_p = (\text{trace}(x^*x)^{p/2})^{1/p} < \infty$. (See [9, Chapter III] for a detailed study of these spaces and related topics.) A matrix $a = (a(i, j))$ is said to be a *Schur-Hadamard multiplier* (or, briefly, a *multiplier*) from C_{p_1} into C_{p_2} if for every $b = (b(i, j))$ in C_{p_1} the Schur-Hadamard product of a and b , namely $a \circ b = (a(i, j)b(i, j))$, belongs to C_{p_2} . We denote by $M(C_{p_1}, C_{p_2})$ the space of all multipliers from C_{p_1} into C_{p_2} with the norm

$$\|a\|_{p_1, p_2} = \sup\{\|a \circ b\|_{p_2}; \|b\|_{p_1} \leq 1\}.$$

In this note we study multipliers of the form

$$a_{f,\lambda} = (a_{f,\lambda}(i, j)),$$

where

$$(1.1) \quad a_{f,\lambda}(i, j) = \begin{cases} \frac{f(\lambda_i) - f(\lambda_j)}{\lambda_i - \lambda_j}, & \lambda_i \neq \lambda_j, \\ f'(\lambda_i), & \lambda_i = \lambda_j. \end{cases}$$

Here f is a continuously differentiable function on $[-1, 1]$ and $\lambda = (\lambda_i)$ is a real sequence with $-1 \leq \lambda_i \leq 1, i = 1, 2, \dots$. Such multipliers are of great importance in perturbation theory of linear operators (see [1, 3, 5-8]). The multiplier $a_{f,\lambda}$ plays the role of the Gâteaux derivative of the operator map $x \mapsto f(x)$, evaluated at the diagonal matrix $d(\lambda) = \text{diag}(\lambda_i)$, whenever the derivative exists.

In the papers [1-5] the authors study general multipliers from C_p into itself (and from $B(l_2)$ into itself) under the name "Stieltjes Double Operator Integrals". In their applications to the multipliers $a_{f,\lambda}$ there is no restriction on λ besides boundedness. As for f , boundedness of the derivative f' is clearly a necessary condition for the boundness of the multiplier $a_{f,\lambda}$ in C_p for all λ , but if $p \neq 2$ it is not sufficient and

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one has to impose more restrictive smoothness conditions on f . (The case $p = 2$ is trivial. C_2 , the Hilbert-Schmidt class, is a Hilbert space by itself and thus its multipliers are precisely the coordinate-wise bounded matrices.) A typical result is the following

THEOREM 1.1 [1, THEOREM 5]. *Let f be so that $f' \in \text{Lip}_\alpha[-1, 1]$ for some $\alpha > 0$, and let $\lambda = (\lambda_i)$ satisfies $-1 \leq \lambda_i \leq 1$ for all i . Then $a_{f,\lambda} \in M(B(l_2), B(l_2))$ and $a_{f,\lambda} \in M(C_p, C_p)$ for all $1 \leq p \leq \infty$.*

We are interested here in the little different problem of membership of $a_{f,\lambda}$ in $M(C_{p_1}, C_{p_2})$ for $p_1 \neq p_2$, mainly for $p_2 \leq 2 \leq p_1$. Roughly speaking, we require that $\lambda \in l_r$ for some $r < \infty$, and relax the Lip_α condition on f' by requiring only that $|f'(t)| \leq C|t|^\alpha$ for some $0 < C < \infty$ and all $-1 \leq t \leq 1$, where the exponent $\alpha > 0$ satisfies $1/p_2 \leq 1/p_1 + \alpha/r$. The condition $p_2 \leq 2 \leq p_1$ allows us to factor $a_{f,\lambda}$ through C_2 , and thus to reduce the study of the multipliers $a_{f,\lambda}$ to that of the multipliers $b_\mu = (\mu_{\min\{i,j\}})$ where $\mu_i = |\lambda_i|^\beta$ for an appropriate β . Finally, using the triangular projection, we reduce the study of the multipliers b_μ to that of left and right multiplication by diagonal matrices. We also show that, in a sense, the result stated in the abstract is best possible.

2. The main result. We start with some known facts on multipliers which will be needed later.

PROPOSITION 2.1. $M(C_2, C_2) = 1_\infty(\mathbb{N} \times \mathbb{N})$.

PROOF. This follows immediately from $C_2 = l_2(\mathbb{N} \times \mathbb{N})$. $\square$

PROPOSITION 2.2. *Let $1 \leq p_1 \leq q_1 \leq \infty$, $1 \leq q_2 \leq p_2 \leq \infty$. Then $M(C_{q_1}, C_{q_2}) \subseteq M(C_{p_1}, C_{p_2})$ and $\|a\|_{p_1, p_2} \leq \|a\|_{q_1, q_2}$ for all $a \in M(C_{q_1}, C_{q_2})$.*

PROOF. This follows trivially from the fact that if $p \leq q$ then $C_p \subseteq C_q$ and $\|x\|_q \leq \|x\|_p$ for all $x \in C_p$. $\square$

COROLLARY 2.3. *Let $1 \leq p_1 \leq 2 \leq p_2 \leq \infty$. Let $\lambda = (\lambda_i)$ be so that $-1 \leq \lambda_i \leq 1$ for all i , and let f be any continuously differentiable function on $[-1, 1]$. Then $a_{f,\lambda} \in M(C_{p_1}, C_{p_2})$.*

PROOF. By Propositions 2.1 and 2.2, $a_{f,\lambda} \in M(C_2, C_2) \subseteq M(C_{p_1}, C_{p_2})$. $\square$

Let D denotes the diagonal projection, i.e. $(Da)(i, j) = \delta_{i,j}a(i, i)$.

PROPOSITION 2.4.

$$D(M(C_{p_1}, C_{p_2})) = M(l_{p_1}, l_{p_2}) = l_{p_0}$$

where $1/p_2 = 1/p_1 + 1/p_0$.

PROOF. l_p is identified with $D(C_p)$, so

$$D(M(C_{p_1}, C_{p_2})) = M(D(C_{p_1}), D(C_{p_2})) = M(l_{p_1}, l_{p_2}).$$

The fact that $M(l_{p_1}, l_{p_2}) = l_{p_0}$ is well known. $\square$

Next for $0 < \alpha < \infty$ let X_α be the space of all continuously differentiable functions on $[-1, 1]$ satisfying $f(0) = 0$ and $|f'(t)| \leq C|t|^\alpha$ for some $0 < C < \infty$ and all $t \in [-1, 1]$, with the norm

$$\|f\|_\alpha = \inf\{0 < C < \infty; |f'(t)| \leq C|t|^\alpha \text{ for all } -1 \leq t \leq 1\}.$$

Our main result is the following:

THEOREM 2.5. *Let $0 < \alpha, r < \infty$. Let $f \in X_\alpha$ and let $\lambda = (\lambda_i) \in l_r$ be so that $-1 \leq \lambda_i \leq 1$ for all i . Then*

$$a_{f,\lambda} = ((f(\lambda_i) - f(\lambda_j))/(\lambda_i - \lambda_j))$$

is a multiplier from C_{p_1} into C_{p_2} for all p_1, p_2 satisfying $1 \leq p_2 \leq 2 \leq p_1 \leq \infty$ and $1/p_2 = 1/p_1 + \alpha/r$.

Moreover,

$$\|a_{f,\lambda}\|_{p_1,p_2} \leq 2\|f\|_\alpha \cdot \|\lambda\|_r^\alpha.$$

We remark that using Proposition 2.2 one gets easily that, under the hypotheses of Theorem 2.5, $a_{f,\lambda} \in M(C_{p_1}, C_{p_2})$ for all $1 \leq p_2 \leq 2 \leq p_1 \leq \infty$ with $1/p_2 \leq 1/p_1 + \alpha/r$.

The proof of Theorem 2.5 proceeds by a sequence of propositions.

PROPOSITION 2.6. *Let $0 < \alpha < \infty$, let $f \in X_\alpha$ and let $\lambda = (\lambda_i)$ be so that $-1 \leq \lambda_i \leq 1$ for all i . Let $m_{f,\lambda,\alpha}$ be defined by*

$$(2.1) \quad m_{f,\lambda,\alpha}(i,j) = \begin{cases} (f(\lambda_i) - f(\lambda_j))/(\lambda_i - \lambda_j) \max\{|\lambda_i|^\alpha, |\lambda_j|^\alpha\}^{-1} & \text{if } \lambda_i \neq 0 \text{ or } \lambda_j \neq 0, \\ 0 & \text{if } \lambda_i = \lambda_j = 0. \end{cases}$$

Then $m_{f,\lambda,\alpha} \in M(C_2, C_2)$ and $\|m_{f,\lambda,\alpha}\|_{2,2} \leq \|f\|_\alpha$.

PROOF. Using Proposition 2.1 we get

$$\|m_{f,\lambda,\alpha}\|_{2,2} = \sup_{i,j} |m_{f,\lambda,\alpha}(i,j)| \leq \|f\|_\alpha. \quad \square$$

Next, let P_T denotes the (upper) triangular projection, that is

$$(P_T x)(i,j) = \begin{cases} x(i,j), & i \leq j, \\ 0, & i > j. \end{cases}$$

It is known that P_T is bounded in C_p if and only if $1 < p < \infty$ (see [10, Chapter III and 11]). In this case, let γ_p denotes the norm of P_T in C_p .

For any $\lambda = (\lambda_i) \in l_\infty$ let $d(\lambda) = \text{diag}(\lambda_i)$ be the diagonal matrix whose (i,i) entry is λ_i . We denote by L_λ and R_λ the operators of left and right multiplication by $d(\lambda)$, respectively. That is

$$(L_\lambda x)(i,j) = \lambda_i x(i,j), \quad (R_\lambda x)(i,j) = x(i,j) \lambda_j.$$

Since $L_\lambda, R_\lambda, P_T$ are (identified with) multipliers—they commute with each other.

Finally, for all sequences $\lambda = (\lambda_i)$ we define a matrix $b_\lambda = (b_\lambda(i,j))$ by

$$b_\lambda(i,j) = \lambda_{\min\{i,j\}}.$$

Notice that for all matrix x

$$(2.2) \quad b_\lambda \circ x = L_\lambda P_T x + R_\lambda (I - P_T) x = P_T L_\lambda x + (I - P_T) R_\lambda x.$$

PROPOSITION 2.7. *Let $1 \leq p_0, p_1, p_2 \leq \infty$ be so that $1/p_2 = 1/p_0 + 1/p_1$. Let $\lambda \in l_{p_0}$. Then L_λ and R_λ map C_{p_1} into C_{p_2} and $\|L_\lambda\|_{p_1,p_2} = \|R_\lambda\|_{p_1,p_2} = \|\lambda\|_{p_0}$.*

We omit the straightforward proof which depends on the “generalized Hölder inequality” in the spaces C_p (see [9, Chapter III]):

$$\|xy\|_{p_2} \leq \|x\|_{p_0} \|y\|_{p_1}, \quad 1/p_2 = 1/p_0 + 1/p_1.$$

PROPOSITION 2.8. *Let $1 \leq p_0, p_1, p_2 \leq \infty$ be so that $1/p_2 = 1/p_0 + 1/p_1$. Let $\lambda \in l_{p_0}$. Then $b_\lambda \in M(C_{p_1}, C_{p_2})$ with norm $\|b_\lambda\|_{p_1, p_2} \leq A\|\lambda\|_{p_0}$ (the constant A depends only on p_1, p_2), except for the cases $p_1 = p_2 = \infty$ and $p_1 = p_2 = 1$.*

PROOF. If $1 < p_1 < \infty$ then P_T is bounded in C_{p_1} , and L_λ and R_λ map C_{p_1} into C_{p_2} . Using (2.2) and Proposition 2.7 we get $b_\lambda \in M(C_{p_1}, C_{p_2})$ and $\|b_\lambda\|_{p_1, p_2} \leq (1 + 2\gamma_{p_1})\|\lambda\|_{p_0}$. (If $1 < p_2 < \infty$ then, using the boundedness of P_T in C_{p_2} , we get in a similar way $b_\lambda \in M(C_{p_1}, C_{p_2})$ and $\|b_\lambda\|_{p_1, p_2} \leq (1 + 2\gamma_{p_2})\|\lambda\|_{p_0}$.) It remains to deal with the case $p_0 = p_2 = 1, p_1 = \infty$. For $x \in C_\infty$ define $x_n(i, j) = x(i, j)$ if $\min\{i, j\} = n$ and $x_n(i, j) = 0$ otherwise. Clearly, $x = \sum_{n=1}^\infty x_n$ and $\|x_n\|_1 \leq 2\|x\|_\infty$. So

$$\begin{aligned} \|b_\lambda \circ x\|_1 &= \left\| \sum_{n=1}^\infty \lambda_n x_n \right\|_1 \leq \sum_{n=1}^\infty |\lambda_n| \|x_n\|_1 \\ &\leq 2\|x\|_\infty \sum_{n=1}^\infty |\lambda_n| = 2\|x\|_\infty \|\lambda\|_1, \end{aligned}$$

i.e. $b_\lambda \in M(C_\infty, C_1)$ and $\|b_\lambda\|_{\infty, 1} \leq 2\|\lambda\|_1$. $\square$

REMARK. For general $\lambda \in l_\infty$, b_λ need not be defined on the whole of C_∞ (or C_1); this is the consequence of [11, Proposition 1.3] and the unboundedness of the triangular projection in C_∞ (respectively, in C_1). If $\lambda \in c_0$ and

$$\sum_{i=1}^\infty \lambda_i^*(2i-1)^{-1} < \infty,$$

where $\{\lambda_i^*\}_{i=1}^\infty$ is the nonincreasing rearrangement of $\{|\lambda_i|\}_{i=1}^\infty$, then

$$b_\lambda \in M(C_\infty, C_\infty) = M(C_1, C_1).$$

This follows by the above arguments from the fact that P_T acts continuously from the Macaev ideal

$$C_\omega = \left\{ x \in C_\infty; \|x\|_\omega = \sum_{i=1}^\infty s_i(x)(2i-1)^{-1} < \infty \right\}$$

into C_∞ and from C_1 into $C_\Omega = C_\omega^*$. (Here $\{s_i(x)\}_{i=1}^\infty$ are the s -numbers of x ; see [10, Chapter III].)

PROOF OF THEOREM 2.5. Since $\lambda_i \rightarrow 0$, we can assume without loss of generality that $\{|\lambda_i|\}_{i=1}^\infty$ is nonincreasing. Thus

$$\max\{|\lambda_i|^\alpha, |\lambda_j|^\alpha\} = |\lambda_{\min\{i, j\}}|^\alpha.$$

Let α_1 be defined by $1/2 = 1/p_1 + \alpha_1/r$ and let $\alpha_2 = \alpha - \alpha_1$. Clearly, $0 \leq \alpha_1, \alpha_2 \leq \alpha$ and $1/p_2 = 1/2 + \alpha_2/r$. We present the proof in the case $p_2 < 2 < p_1$, i.e. $0 < \alpha_1, \alpha_2 < \alpha$. The other cases are treated similarly, and are even easier.

Let $m_{f, \lambda, \alpha}$ be defined by (2.1), let $\mu^{(k)} = |\lambda|^{\alpha_k}$ ($k = 1, 2$), i.e. $\mu_i^{(k)} = |\lambda_i|^{\alpha_k}$. Then $a_{f, \lambda}$ admits the following factorization:

$$a_{f, \lambda} = b_{\mu^{(2)}} \circ m_{f, \lambda, \alpha} \circ b_{\mu^{(1)}}.$$

Now, by Proposition 2.8 $b_{\mu(1)} \in M(C_{p_1}, C_2)$, $b_{\mu(2)} \in M(C_2, C_{p_2})$ and

$$\begin{aligned} \|b_{\mu(1)}\|_{p_1, 2} &\leq \sqrt{2} \|\mu^{(1)}\|_{r/\alpha_1} = \sqrt{2} \|\lambda\|_r^{\alpha_1}, \\ \|b_{\mu(2)}\|_{2, p_2} &\leq \sqrt{2} \|\mu^{(2)}\|_{r/\alpha_2} = \sqrt{2} \|\lambda\|_r^{\alpha_1}. \end{aligned}$$

Since, by Proposition 2.6, $m_{f, \lambda, \alpha} \in M(C_2, C_2)$ and $\|m_{f, \lambda, \alpha}\|_{2, 2} \leq \|f\|_\alpha$ we finally get that $a_{f, \lambda} \in M(C_{p_1}, C_{p_2})$ and

$$\begin{aligned} \|a_{f, \lambda}\|_{p_1, p_2} &\leq \|b_{\mu(2)}\|_{2, p_2} \cdot \|m_{f, \lambda, \alpha}\|_{2, 2} \cdot \|b_{\mu(1)}\|_{p_1, 2} \\ &\leq \sqrt{2} \|\lambda\|_r^{\alpha_2} \cdot \|f\|_\alpha \cdot \sqrt{2} \|\lambda\|_r^{\alpha_1} = 2 \|f\|_\alpha \|\lambda\|_r^\alpha. \quad \square \end{aligned}$$

3. Concluding remarks. Theorem 2.5 admits the following partial converse, which shows that our hypotheses on f and λ are optimal.

PROPOSITION 3.1. *Let $1 \leq p_2 < p_1 \leq \infty$, let $\alpha > 0$ and let $0 < r < \infty$ be so that $1/p_2 = 1/p_1 + \alpha/r$.*

- (i) *Let $\lambda = (\lambda_i)$ be so that $-1 \leq \lambda_i \leq 1$ for all i , and assume that for all $f \in X_\alpha$, $a_{f, \lambda} \in M(C_{p_1}, C_{p_2})$. Then $\lambda \in l_r$.*
- (ii) *Let f be a continuous differentiable function on $[-1, 1]$ so that for $\lambda = (\lambda_i) \in l_r$ with $-1 \leq \lambda_i \leq 1$ for all i , $a_{f, \lambda} \in M(C_{p_1}, C_{p_2})$. Then $f \in X_\alpha$.*

PROOF. (i) Consider $f(t) = |t|^\alpha$. We have $f'(t) = (\alpha + 1)|t|^\alpha$ and so $f \in X_\alpha$. Since $a_{f, \lambda} \in M(C_{p_1}, C_{p_2})$, we get by Proposition 2.4 that

$$D(a_{f, \lambda}) = \text{diag}((\alpha + 1)(\lambda_i)^\alpha) \in M(l_{p_1}, l_{p_2}) = l_{r/\alpha}.$$

That is $\lambda \in l_r$.

(ii) Suppose that $f \notin X_\alpha$. Then there is a sequence $\{t_n\}_{n=1}^\infty$ in $[-1, 1]$ with $0 < |t_n| \leq 2^{-n(1+1/\alpha)}$ and $|f'(t_n)| \geq 2^n |t_n|^\alpha$ for all $n = 1, 2, \dots$. Put

$$k_n = [2^{-nr/\alpha} \cdot |t_n|^{-r}], \quad n = 1, 2, \dots,$$

and let $\lambda = (\lambda_i)$ be the sequence in which t_n appears exactly k_n times. Then

$$\sum_{i=1}^\infty |\lambda_i|^r = \sum_{n=1}^\infty k_n |t_n|^r \leq \sum_{n=1}^\infty 2^{-nr/\alpha} < \infty,$$

but

$$\begin{aligned} \sum_{i=1}^\infty |f'(\lambda_i)|^{r/\alpha} &= \sum_{n=1}^\infty k_n |f'(t_n)|^{r/\alpha} \\ &\geq \sum_{n=1}^\infty k_n 2^{nr/\alpha} |t_n|^r \geq \sum_{n=1}^\infty (1 - 2^{-nr}) = \infty. \end{aligned}$$

It follows that $(f'(\lambda_i)) \notin M(l_{p_1}, l_{p_2})$, and so by Proposition 2.4,

$$a_{f, \lambda} \notin M(C_{p_1}, C_{p_2}).$$

A contradiction. $\square$

REMARK. Our methods and results can be extended by standard arguments to other symmetric norm ideals. For instance, if $a \in M(C_{p_1}, C_{p_2})$ then $a \in M(C_{p_2^*}, C_{p_1^*})$ where $1/p_j + 1/p_j^* = 1$, $j = 1, 2$. This is the consequence of the fact that the adjoint operator of the multiplier a is the multiplier $\bar{a} = (\overline{a(i, j)})$, the fact that $\|\bar{a}\|_{p, q} = \|a\|_{p, q}$ and the well-known fact that $(C_p)^* = C_{p^*}$. It follows

that if $\mathcal{F}$ is any interpolation functor and if $C_E = \mathcal{F}(C_{p_1}, C_{p_2}) = C_{\mathcal{F}(l_{p_1}, l_{p_2}^*)}$, $C_F = \mathcal{F}(C_{p_2}, C_{p_1}^*) = C_{\mathcal{F}(l_{p_2}, l_{p_1}^*)}$ then $a \in M(C_E, C_F)$. For instance, if $0 < \theta < 1$ and $p(\theta), q(\theta)$ are defined by $1/p(\theta) = (1-\theta)/p_1 + \theta/p_2^*$ and $1/q(\theta) = (1-\theta)/p_2 + \theta/p_1^*$, then $a \in M(C_{p(\theta)}, C_{q(\theta)})$.

We conclude the paper with the following

CONJECTURE. *Theorem 2.5 remains true even in the case where p_1, p_2 are on the same side of 2, i.e., $1 \leq p_2 \leq p_1 \leq 2$, or $2 \leq p_2 \leq p_1 \leq \infty$.*

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