ON ZERO SETS FOR THE BALL ALGEBRA¹

H. ALEXANDER

ABSTRACT. Rudin has constructed a function in the ball algebra in $\mathbb{C}^2$ whose zero set has infinite volume and has asked if such functions exist in $\mathbb{C}^n$. Using some recent work of Ryll and Wojtaszczyk on homogeneous polynomials we shall extend Rudin's proof to $\mathbb{C}^n$ for arbitrary n .

1. Introduction. Let $A(B_n)$ denote the ball algebra, that is, the space of functions holomorphic on the open unit ball B_n in $\mathbb{C}^n$ and continuous on its closure. For $n = 2$, Rudin [1, p. 139] has given an example of a function f in the ball algebra whose zero set in B_2 has infinite volume (= area, in this case). His method involves a construction with monomials in one complex variable and Rudin has asked [1, pp. 139, 414] if such a result holds for $n > 2$, even for an H^∞ function.

Recently Ryll and Wojtaszczyk [2] have proved the existence of homogeneous polynomials in n complex variables whose L^2 and L^∞ norms on ∂B_n are comparable, independent of their degrees. Jointly in [2] and also in [3] they have shown that certain results in one complex variable can be extended to several complex variables by replacing monomials in the one-variable constructions with their special homogeneous polynomials in the several variable case. The object of this note is to apply this approach to Rudin's question. We shall adapt Rudin's proof for $n = 2$ to prove:

THEOREM. *For all n , there exists an $f \in A(B_n)$ whose zero set in B_n has infinite $2n - 2$ dimensional volume.*

Professor Rudin has informed me that E. Amar has obtained a like result by other methods.

2. A construction in B_n . According to Ryll and Wojtaszczyk [2] there exist homogeneous polynomials f_k in $\mathbb{C}^n$ of degree k for $k = 1, 2, 3, \dots$, such that $\|f_k\|_\infty = 1$ and $\|f_k\|_2 \geq q$ where the norms are the $L^\infty(\sigma)$ and $L^2(\sigma)$ norms for σ the normalized invariant measure on ∂B_n and $q > 0$ depends only on n , not on k .

Choose a_j , $j = 1, 2, 3, \dots$ such that

(a) $q/4 < a_j < q/3$ and

(b) a_j is a regular value for f_j ; i.e. $df_j \neq 0$ on $\{f_j = a_j\}$.

Now define, for $z \in B_n$,

$$P(z) = \prod_{j=1}^{\infty} \left(1 - \frac{f_{2^j}}{a_{2^j}}\right).$$

Then $P(0) = 1$ and P has multiplicity one on its zero set by (b). From $\|f_j\|_\infty = 1$ it follows that $|f_{2^j}(z)| \leq \|z\|^{2^j}$ for all z , where $\|z\|$ is the Euclidean norm of z .

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This and $1/a_{2^j} < 4/q$ imply

$$|P(z)| \leq \prod_{j=1}^{\infty} \left(1 + \left(\frac{4}{q}\right) \|z\|^{2^j}\right) \leq 1 + \sum_{t=1}^{\infty} \left(\frac{4}{q}\right)^{c(t)} \|z\|^t$$

where $c(t)$ is the number of ones in the binary representation of t (this argument follows Rudin [1, p. 138]). From $2^{c(t)} \leq t + 1$ we get

$$|P(z)| \leq 1 + \sum_{t=1}^{\infty} (t+1)^N \|z\|^t \leq \frac{\text{constant}}{(1 - \|z\|)^{N+1}}$$

where N is an integer satisfying $N > \log(4/q)/\log 2$. Now define f on B_{n+1} by

$$f(z, z_{n+1}) = z_{n+1}^{2N+3} P(z)$$

with $z = (z_1, z_2, \dots, z_n)$. Since $|z_{n+1}| \leq (1 - \|z\|^2)^{1/2}$ in B_{n+1} , the last estimate for $P(z)$ implies that f is in the ball algebra $A(B_{n+1})$. We shall prove the theorem by showing that the zero set of f in B_{n+1} has infinite $2n$ dimensional volume.

3. The growth of the zero set of $P(z)$. For $r < 1$ and F holomorphic on B_n we set

$$N(F, r) = \int \log |F(r\alpha)| d\sigma(\alpha)$$

and put, for $\alpha \in \partial B_n$,

$$N(F, r, \alpha) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |F(re^{i\theta}\alpha)| d\theta.$$

Then $N(F, r) = \int N(F, r, \alpha) d\sigma(\alpha)$. We have

$$N(P, r) = \sum_{j=1}^{\infty} N\left(1 - \frac{f_{2^j}}{a_{2^j}}, r\right) = \sum_{j=1}^{\infty} \int N\left(1 - \frac{f_{2^j}}{a_{2^j}}, r, \alpha\right) d\sigma(\alpha).$$

The following fact is standard.

LEMMA. $1/(2\pi) \int \log |1 - \zeta \lambda^n| d\theta = \log^+ |\zeta|$ where $\lambda = e^{i\theta}$, $n = 1, 2, 3, \dots$

This gives

$$\begin{aligned} N\left(1 - \frac{f_{2^j}}{a_{2^j}}, r, \alpha\right) &= \int_{-\pi}^{\pi} \log \left|1 - \frac{r^{2^j} f_{2^j}(\alpha)}{a_{2^j}} \cdot e^{i2^j\theta}\right| \frac{d\theta}{2\pi} \\ &= \log^+ \left(\frac{r^{2^j} |f_{2^j}(\alpha)|}{a_{2^j}}\right). \end{aligned}$$

Hence

$$\begin{aligned} N(P, r) &= \sum_{j=1}^{\infty} \int \log^+ \left(\frac{r^{2^j} |f_{2^j}(\alpha)|}{a_{2^j}}\right) d\sigma(\alpha) \\ &\geq \sum_{j=1}^{\infty} \int_{\{\alpha: |f_{2^j}(\alpha)| \geq 2a_{2^j}/r^{2^j}\}} \log^+ \left(\frac{r^{2^j} |f_{2^j}(\alpha)|}{a_{2^j}}\right) d\sigma(\alpha) \\ &\geq \log 2 \sum_{j=1}^{\infty} \sigma\left\{\alpha: |f_{2^j}(\alpha)| \geq \frac{2a_{2^j}}{r^{2^j}}\right\}. \end{aligned}$$

LEMMA. For $0 < b < q$ and for all k ,

$$\sigma\{\alpha: |f_k(\alpha)| \geq b\} \geq \frac{q^2 - b^2}{1 - b^2} \equiv \delta(b) > 0.$$

PROOF. Let $E_k = \{\alpha: |f_k(\alpha)| \geq b\}$. Then

$$q^2 \leq \int |f_k|^2 d\sigma = \int_{E_k} + \int_{\partial B_n \setminus E_k} \leq \sigma(E_k) + b^2(1 - \sigma(E_k)).$$

For $0 < r < 1$ fixed, consider those j 's such that $2^j \leq \log(5/4)/\log(1/r)$. There are at least

$$J(r) \equiv \log\left(\frac{\log 5/4}{\log 1/r}\right) / \log 2 - 1$$

such j 's. Then $(1/r)^{2^j} \leq 5/4$ and so $2a_{2^j}/r^{2^j} \leq 2 \cdot 5/4 \cdot q/3 = 5q/6$. The previous lemma implies

$$\sigma\left\{|f_{2^j}| \geq \frac{2a_{2^j}}{r^{2^j}}\right\} \geq \sigma\left\{|f_{2^j}| \geq \frac{5}{6}q\right\} \geq \delta\left(\frac{5q}{6}\right) > 0.$$

We get $N(P, r) \geq J(r)\delta(5q/6)$. Because $J(r) \sim \log(1/(1-r))/\log 2$ as $r \rightarrow 1$, it follows that $N(P, r) \rightarrow \infty$ as $r \rightarrow 1$; hence

$$\sup_{0 < r < 1} \int \log |P(r\alpha)| d\sigma(\alpha) = \infty.$$

4. The zero set of f . Let $V = Z(P) \equiv \{z \in B_n: P(z) = 0\}$. Let π_{n+1} be the projection from B_{n+1} to B_n : $\pi_{n+1}(z, z_{n+1}) = z$ for $z \in B_n$. Let $\tilde{V} = \pi_{n+1}^{-1}(V) = \{(z, z_{n+1}) \in B_{n+1}: z \in V\}$. Then recalling $f(z, z_{n+1}) = z_{n+1}^{2N+3}P(z)$ we have $Z(f) = \tilde{V} \cup W$ where $W = \{(z, 0): z \in B_n\}$.

LEMMA. Let A be an analytic subvariety of B_n of pure dimension d and let $\pi_{n+1}^{-1}(A)$ be the corresponding $d+1$ dimensional subvariety of B_{n+1} . Then

$$(2d+2) - \text{vol}(\pi_{n+1}^{-1}(A)) = \pi \int_A (1 - \|z\|^2) dv(z)$$

where v is the $2d$ volume measure on A .

PROOF. For $d+1$ dimensional subvarieties of $\mathbb{C}^n$ the volume form is given by

$$\frac{1}{(d+1)!} \left(\frac{i}{2} \partial \bar{\partial} \|z\|^2 + \frac{i}{2} \partial \bar{\partial} |z_{n+1}|^2 \right)^{d+1}.$$

On varieties of the type $\pi_{n+1}^{-1}(A)$ this reduces to

$$\begin{aligned} \frac{1}{(d+1)!} \left[(d+1) \left(\frac{i}{2} \partial \bar{\partial} \|z\|^2 \right)^d \wedge \frac{i}{2} \partial \bar{\partial} |z_{n+1}|^2 \right] \\ = \left(\frac{1}{d!} \left(\frac{i}{2} \partial \bar{\partial} \|z\|^2 \right)^d \right) \wedge \frac{i}{2} \partial \bar{\partial} |z_{n+1}|^2. \end{aligned}$$

This last expression is the product measure on $\pi_{n+1}^{-1}(A)$ and, since the fibre of $\pi_{n+1}^{-1}(A)$ over a point $z \in A$ is a disc of radius $(1 - \|z\|^2)^{1/2}$, the result follows by Fubini's theorem.

It is known (cf. [1, §17.3.6]) that there are several equivalent ways to formulate the Blaschke condition in the ball B_n . In particular, the conditions

$$(a) \quad \sup_{0 < r < 1} \int_{\partial B_n} \log |P(r\alpha)| d\sigma(\alpha) < \infty$$

and

$$(b) \quad \int_{Z(P)} (1 - \|z\|^2) dv < \infty$$

are equivalent. We have shown that (a) does not hold. Hence the integral in (b) does not converge. By the previous lemma we conclude that the $2n$ -volume of $\tilde{V} = \pi_{n+1}^{-1}(V)$ is infinite and consequently that the zero set of f in B_{n+1} has infinite volume. This completes the proof of our theorem.

REMARK. Implicit in this argument is the fact that the previous lemma implies that, for a subvariety V of B_n with $\tilde{V} = \pi_{n+1}^{-1}(V)$, $\tilde{V}$ in B_{n+1} has finite volume if and only if V satisfies the Blaschke condition in B_n .

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS AT CHICAGO CIRCLE, BOX 4348, CHICAGO, ILLINOIS 60680