PROPER PSEUDOCOMPACT EXTENSIONS OF COMPACT ABELIAN GROUP TOPOLOGIES¹

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ABSTRACT. A compact Abelian group G admits a strictly finer pseudocompact topological group topology if and only if the weight of G is uncountable.

1. Introduction and motivation. Many authors have considered questions concerning increasing or reducing the topology of a topological group without disturbing certain specific features. For example: Can the topology of a locally compact group be enlarged to a locally compact group without augmenting the family of closed subgroups [12, 11, 10]? Does every (Hausdorff) group, if not totally bounded, admit a coarser Hausdorff topological group topology [11, 14, 6]? Does every totally bounded topological group topology extend to a strictly larger one [2]?

The present paper is a contribution in this vein. Motivated by questions in topological Galois theory [13, 15], where pseudocompact groups play a basic rôle [16], and by a fascination with pseudocompact groups as a subject of interest in its own right [3, 20, 21, 1], we show that every compact Abelian group with uncountable weight does admit a strictly finer pseudocompact group topology. Our point of departure is [4], where the question we consider was raised, and answered for totally disconnected groups.

The referee has pointed out that there is a close connection between our result and Theorem 4.4 of Hewitt and Ross [8]. A sharpening of the technique of [8] can be used to prove a result much stronger than that of the present paper, and to demonstrate the existence of finer pseudocompact group topologies of (maximal) weight 2^{2^α} when G is compact Abelian with $w(G) = \alpha > \omega$. We intend to discuss this in another paper.

In an attempt to make this work reasonably self-contained, we record from [4] those arguments necessary for a complete proof of our theorem.

2. Definitions; Results from the literature. All the topological groups hypothesized in this paper are assumed to satisfy the T_0 separation property; as is well known (see for example [7, Theorem 8.4]), this guarantees that they are in fact completely regular, Hausdorff spaces, i.e., Tychonoff spaces.

A topological group G is said to be *totally bounded* if for every nonempty, open subset U of G there is finite $F \subset G$ such that $G = FU$. It is a theorem of Weil

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[19] that every totally bounded group G embeds as a dense subgroup of a compact group $\bar{G}$; further, $\bar{G}$ is unique in the sense that if $\tilde{G}$ is a compact group in which G is dense then there is a function φ , simultaneously an algebraic isomorphism and a topological homeomorphism, from $\bar{G}$ onto $\tilde{G}$, such that $\varphi(x) = x$ for all $x \in G$. The group $\bar{G}$ is called *the Weil completion* of the (totally bounded) group G .

A topological space X is said to be *pseudocompact* if every continuous, real-valued function on X has bounded range; for Tychonoff spaces this condition is equivalent to the condition that every locally finite family of nonempty open subsets of X is finite. It is not difficult to show that every pseudocompact group G is totally bounded [3] (and hence is dense in the compact group $\bar{G}$). The following result from [3] helps to explain which totally bounded groups are pseudocompact.

2.1. Let K be a compact group and G a dense subgroup of K . Then G is pseudocompact if and only if G is G_δ -dense in K in the sense that every nonempty G_δ of K has nonempty intersection with G .

2.2. LEMMA. Let G and $\tilde{G}$ be compact groups, φ a continuous homomorphism of G onto $\tilde{G}$, and E a nonempty G_δ in G . Then φ is an open function, and $\varphi[E]$ contains a nonempty (compact) G_δ in $\tilde{G}$.

PROOF. That φ is open follows from the Baire category theorem. The set E contains a set of the form $\bigcap_n U_n$, with $\text{cl}_G U_{n+1} \subset U_n$ and U_n open in G for $n < \omega$. The set $\bigcap_n \varphi[U_n]$, which is $\bigcap_n \varphi[\text{cl}_G U_n]$, is then as required.

We remark in passing that (as is shown in [4]) it follows without difficulty from (2.2) that if G and $\tilde{G}$ are compact groups, φ a continuous homomorphism from G onto $\tilde{G}$ and $\tilde{H}$ a dense, pseudocompact subgroup of $\tilde{G}$, then $\varphi^{-1}(\tilde{H})$ is a dense, pseudocompact subgroup of G . In certain contexts this observation makes it easy to define dense, pseudocompact subgroups of a given compact group G . Since the result is not needed here, we omit the details of the proof.

In addition to the usual elements of Pontrjagin duality, as set forth, for example, in [7], we need the following algebraic results concerning the p -group decomposition of an Abelian torsion group.

We denote by P the set of prime numbers.

2.3. (See [7, A.3]). Let G be an Abelian torsion group and for $p \in P$ let G_p be the set of elements of G whose order is a power of p . Then G_p is a p -primary subgroup of G , and G is isomorphic to the group $\bigoplus_{p \in P} G_p$.

2.4. (See [7, A.11(III)].) Let G be an Abelian p -group with socle $S = \{x \in G: \text{order of } x = p\} \cup \{e\}$, for $x \in G$ let $p^{k(x)}$ be the order of x in G and let L be a maximal independent set in G . Then $\{p^{k(x)-1}x: x \in L\}$ is a basis for S .

2.5. COROLLARY. Let G be an Abelian p -group with $|G| > \omega$ and let S be the socle of G . Then $|S| = |G|$.

PROOF. With L as in 2.4 we have $|G| \leq |L| \cdot \omega$ and $|S| \geq |L|$; from $|G| > \omega$ follows $|S| = |G|$, as required.

For a space X and $x \in X$, the symbols $w(X)$ and $\chi(x, X)$ denote respectively the weight of X and the local weight of X at x .

The following statements can be substantially generalized, but they are adequate to our needs in their present form.

2.6. (See [7, Theorem 24.15 and 9, 28.58(c)].) Let G be a compact Abelian group with dual group $\hat{G}$. Then $\chi(e, G) = w(G) = |\hat{G}|$.

3. Proof of the theorem. This first, elementary lemma is a special case of a general topological result noticed by R. M. Stephenson, Jr. [17]; the present proof is taken from [4].

3.1. LEMMA. *Let $G = \langle G, \tau \rangle$ be a compact group with $w(G) \leq \omega$ and let τ' be a pseudocompact group topology for G such that $\tau' \supset \tau$. Then $\tau' = \tau$.*

PROOF. Each point of G is a $\tau' - G_\delta$ of G and hence of the Weil completion $\bar{G}$ of $\langle G, \tau' \rangle$. It follows that every point of $\bar{G}$ is a G_δ . Thus $\chi(p, \bar{G}) \leq \omega$ for all $p \in \bar{G}$ and hence $\bar{G}$ is metrizable (cf. [7, Theorem 8.3]). Thus $\langle G, \tau' \rangle$ is a pseudocompact metric space, hence is compact (see for example [5, Exercise 3D.2]). The identity function from $\langle G, \tau' \rangle$ onto $\langle G, \tau \rangle$ is then a homeomorphism, as required.

For topological groups G and $\tilde{G}$, we write $G \cong \tilde{G}$ if some function from G onto $\tilde{G}$ is both an isomorphism and a topological homeomorphism.

In what follows we denote by $\mathbf{Z}$ and $\mathbf{T}$ the set of integers and the circle, each endowed with the usual algebraic and topological properties. For $p \in P$ the (additive) group of integers mod p is denoted $\mathbf{Z}(p)$, and for $0 < n < \omega$ we write $\mathbf{T}(n) = \{t \in \mathbf{T} : t^n = 1\}$.

DEFINITION. For a cardinal number α , a subgroup K of $\mathbf{T}^\alpha$ is a **T-elementary subgroup** of $\mathbf{T}^\alpha$ if there is a closed subgroup S of $\mathbf{T}$, with $|S| > 1$, such that $K = S^\alpha$.

We note that $\mathbf{T}^\alpha$ is itself T-elementary in $\mathbf{T}^\alpha$. Indeed, the T-elementary subgroups of $\mathbf{T}^\alpha$ are exactly $\mathbf{T}^\alpha$ and $\mathbf{T}(n)^\alpha$ for $0 < n < \omega$.

3.2. LEMMA. *Let G be a compact, Abelian group with $w(G) > \omega$. Then*

- (a) *there is a subgroup H of $\hat{G}$ such that either $H \cong \bigoplus_{i \in I} \mathbf{Z}_i$ with $|I| > \omega$ or $H \cong \bigoplus_{i \in I} \mathbf{Z}(p)_i$ (for some $p \in P$) with $|I| > \omega$; and*
- (b) *there exist $\alpha > \omega$ and a T-elementary subgroup K of $\mathbf{T}^\alpha$ and a continuous homomorphism of G onto K .*

PROOF. (a) Let r_0 and r_p ($p \in P$) denote respectively the torsion-free rank and the p -rank, and let M denote the injective hull (that is, the minimal divisible extension) of $\hat{G}$. Using (2.6) and [7, Theorems A.14 and A.16] we have

$$\begin{aligned} \omega < w(G) = |\hat{G}| &\leq |M| = \omega \left[r_0(M) + \sum_{p \in P} r_p(M) \right] \\ &= \omega \left[r_0(\hat{G}) + \sum_{p \in P} r_p(\hat{G}) \right]; \end{aligned}$$

it follows that either $r_0(\hat{G}) > \omega$ or there is $p \in P$ such that $r_p(\hat{G}) > \omega$. In the first case we take

$$H = \bigoplus_{i \in I} \mathbf{Z}_i \quad \text{with } |I| = r_0(\hat{G});$$

and in the second case, with H the socle of $\hat{G}_p$, we have

$$\hat{G} \supset H \cong \bigoplus_{i \in I} \mathbf{Z}(p)_i \quad \text{with } |I| = r_p(\hat{G}).$$

- (b) The annihilator A of H in G , defined by the rule

$$A = \{x \in G : \chi \in H \text{ implies } \chi(x) = 1\},$$

satisfies $G/A \cong \hat{H}$ (see [7, Theorem 24.11]). When $H \cong \bigoplus_{i \in I} \mathbb{Z}_i$ we set

$$K = T^I = \prod_{i \in I} (\mathbb{Z}_i)^\wedge = \hat{H},$$

and when $H \cong \bigoplus_{i \in I} \mathbb{Z}(p)_i$ we set

$$K = T(p)^I = \prod_{i \in I} (\mathbb{Z}(p)_i)^\wedge = \hat{H};$$

the canonical homomorphism: $G \rightarrow G/A \cong \hat{H} = K$ is then as required.

3.3. LEMMA. *Let G be a compact, Abelian group and suppose there is a continuous homomorphism φ of G onto a T -elementary subgroup $K = S^\alpha \subset T^\alpha$ with $\alpha > \omega$. Then there is homomorphism Ω of G onto S such that the graph of Ω is a dense, pseudocompact subgroup of $G \times S$.*

PROOF. We set $T_\xi = T$ for $\xi < \alpha$, and

$$W = \bigoplus_{\xi < \alpha} T_\xi \subset T^\alpha.$$

We define a homomorphism ψ from W to T by the rule $\psi(t) = \prod_{\xi < \alpha} t_\xi$ (this is a finite product for each $t \in W$) and, appealing to the divisibility of T , we choose any homomorphism $\tilde{\psi}: T^\alpha \rightarrow T$ such that $\tilde{\psi}|_W = \psi$; we define $\bar{\psi} = \tilde{\psi}|_K$.

It is clear that $\bar{\psi}[K] \supset S$ —indeed, $\bar{\psi}[K \cap W] \supset S$. We claim $\bar{\psi}[K] = S$. If $S = T$ this is clear, and if $S = T(n)$ then for $t \in K$ we have $t^n = 1 \in T^\alpha$ and hence $(\bar{\psi}(t))^n = 1 \in T$, i.e., $\bar{\psi}(t) \in S$.

Now we define $\Omega = \bar{\psi} \circ \varphi: G \rightarrow S$ and we set $H = \text{graph } \Omega \subset G \times S$. It is clear that H is a subgroup of the compact group $G \times S$. To complete the proof it is, by (2.1), enough to show that H is G_δ -dense in $G \times S$.

Let F be a nonempty G_δ of $G \times S$. The set F contains a set of the form $E \times \{p\}$ with E a nonempty G_δ of G and with $p \in S$, and from (2.2) (with $\tilde{G}$ replaced by K) it follows that there is a nonempty G_δ D of K such that $D \subset \varphi[E]$. The set D contains a set of the form

$$\{t(C)\} \times S^{\alpha \setminus C} \subset S^\alpha = K$$

with C a countable subset of α and with $t(C) \in S^C$.

Now define $\bar{t} \in K$ by

$$\begin{aligned} \bar{t}_\xi &= t(C)_\xi & \text{for } \xi \in C \\ &= 1 & \text{for } \xi \in \alpha \setminus C, \end{aligned}$$

set $q = \bar{\psi}(\bar{t}) \in S$, choose $\eta \in \alpha \setminus C$ and define $\bar{s} \in K$ by

$$\begin{aligned} \bar{s}_\xi &= \bar{t}_\xi = t(C)_\xi \in S & \text{for } \xi \in C \\ &= pq^{-1} \in S & \text{for } \xi = \eta \\ &= 1 \in S & \text{for } \xi \in \alpha \setminus (C \cup \{\eta\}). \end{aligned}$$

Then

$$\bar{\psi}(\bar{s}) = \bar{\psi}(\bar{t} \cdot \bar{t}^{-1} \cdot \bar{s}) = q \cdot \psi(\bar{t}^{-1} \cdot \bar{s}) = q \cdot pq^{-1} = p.$$

Since $\bar{s} \in \{t(C)\} \times S^{\alpha \setminus C} \subset D \subset \varphi[E]$ there is $\bar{x} \in E$ such that $\varphi(\bar{x}) = \bar{s}$, and from

$$\Omega(\bar{x}) = \bar{\psi}(\varphi(\bar{x})) = \bar{\psi}(\bar{s}) = p$$

we have $(\bar{x}, \Omega(\bar{x})) \in H \cap (E \times \{p\}) \subset H \cap F$, as required. The proof is complete.

The theorem stated in the abstract is now easily proved.

3.4 THEOREM. *Let $G = \langle G, \tau \rangle$ be a compact Abelian group. There is a pseudocompact group topology τ' for G with $\tau' \supsetneq \tau$ if and only if $w(G) > \omega$.*

PROOF. The "only if" statement is Lemma 3.1. For the "if" statement we note from Lemmas 3.2 and 3.3 that there is a homomorphism Ω of G onto a closed subgroup S of $\mathbf{T}$ with $|S| > 1$ such that the set $H = \text{graph } \Omega$ is a dense, pseudocompact subgroup of $G \times S$.

Let π be the (continuous) projection from $G \times S$ onto G and τ' the topology on G induced by $\pi|H$, that is, the topology for G defined by the requirement that $\pi|H$ is a homeomorphism of H onto G . That $\tau' \supset \tau$ follows from the fact that π is continuous (from H onto $\langle G, \tau \rangle$). The containment is proper because $\langle G, \tau \rangle$ is compact while H , a proper, dense subspace of $G \times S$, is not compact.

The proof is complete.

4. Concluding remarks. We address Lemma 3.3 and its proof, retaining the notation used there.

4.1. The homomorphism $\psi: W \rightarrow \mathbf{T}$ given by $\psi(t) = \prod_{\xi < \alpha} t_\xi$ is not continuous. Indeed if ψ were continuous then some extension $\tilde{\psi}$ from $\mathbf{T}^\alpha$ to $\mathbf{T}$ would be continuous and then H , the graph of Ω , would be closed in $G \times S$; this is absurd since H is dense in $G \times S$ and $H \neq G \times S$. Let us note a direct proof that ψ is not continuous. Let r be any element of S such that $r \neq 1$, let $\{\xi(n): n < \omega\}$ be a strictly increasing sequence of predecessors of α , and define $t(n) \in W$ by

$$\begin{aligned} t(n)_\xi &= r \quad \text{for } \xi = \xi(n) \\ &= 1 \quad \text{for } \xi < \alpha, \xi \neq \xi(n). \end{aligned}$$

The sequence $t(n)$ approaches the element of W whose coordinates are all 1, and

$$\lim_n \psi(t(n)) = \lim_n r = r \neq 1 = \psi\left(\lim_n t(n)\right).$$

4.2. The proof of Lemma 3.3 makes it clear that the graph of $\tilde{\psi}$ is G_δ -dense in $\mathbf{T}^\alpha \times \mathbf{T}$; this condition is equivalent to the statement that if C is countable subset of α and $t(C) \in \mathbf{T}^C$ and $p \in \mathbf{T}$, then there is $\bar{s} \in \mathbf{T}^\alpha$ such that $\bar{s}_\xi = t(C)_\xi$ for $\xi \in C$ and $\tilde{\psi}(\bar{s}) = p$. In fact, however, a much stronger statement is true: A value for $\tilde{\psi}$ specified in advance is achieved not only at a point of $\mathbf{T}^\alpha$ for which countably many coordinates are specified in advance, but indeed at some point of $\mathbf{T}^\alpha$ all but one of whose coordinates are specified. The proof is essentially as before: if $\eta < \alpha$ and $t \in \mathbf{T}^{\alpha \setminus \{\eta\}}$ and $p \in \mathbf{T}$, and if we define $\bar{t}$ and $\bar{s} \in \mathbf{T}^\alpha$ by

$$\begin{aligned} \bar{t}_\xi &= \bar{s}_\xi = t_\xi \quad \text{for } \xi \neq \eta, \\ \bar{t}_\eta &= 1, \quad \bar{s}_\eta = p \cdot \bar{\psi}(\bar{t})^{-1}, \end{aligned}$$

then $\tilde{\psi}(\bar{s}) = p$. Algebraically, we have for each $\eta < \alpha$ the isomorphism

$$(\text{graph } \tilde{\psi}) \times \mathbf{T}_\eta = \mathbf{T}^\alpha \times \mathbf{T}.$$

4.3. The fact that $H = \text{graph } \Omega$ is G_δ -dense in $G \times S$ shows (since $\{1\}$ is a G_δ of S) that if E is a nonempty G_δ of G then $(\text{kernel } \Omega) \cap E \neq \emptyset$. That is, the discontinuous homomorphism $\Omega: G \rightarrow \mathbf{T}$ has these properties: Its graph yields a pseudocompact refinement (a covering) of $\langle G, \tau \rangle$, and its kernel is a dense, pseudocompact subgroup of $\langle G, \tau \rangle$.

4.4. It is known [3] that a topological group is pseudocompact if and only if it is totally bounded and its Weil completion coincides with its Stone-Čech compactification. It follows that for $\langle G, \tau \rangle$ as in Theorem 3.4 the Stone-Čech compactification of $\langle G, \tau' \rangle$ is isomorphic to $\langle G, \tau \rangle \times S$. When G is itself a T-elementary subgroup $S^\alpha \subset T^\alpha$ with $\alpha > \omega$, the finer pseudocompact topology τ' we have constructed for G then has the property that the Stone-Čech compactification of $\langle G, \tau' \rangle$ is topologically isomorphic to $S^\alpha \times S$ in its usual topology, i.e., to G itself.

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