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# INTEGRAL POLYNOMIAL GENERATORS FOR THE HOMOLOGY OF $BSU$

STANLEY O. KOCHMAN[1]

ABSTRACT. Explicit formulas are given for polynomial generators of $H_*BSU$ as specific polynomials in the canonical polynomial generators of $H_*BU$. The method is also applied to $H_*(BSU;R)$ for any coefficient ring $R$ and to $H_*(BSO;Z_2)$.

**1. Introduction.** Let $R$ be a commutative ring with unit, and let $A$ be the polynomial algebra $R[X_1,\ldots,X_n,\ldots]$ with $\deg X_n = n\alpha$, $\alpha > 0$, and $X_0 = 1$. Let $\Gamma = \bigoplus_{k=0}^{\infty} R\gamma_k$ be the divided polynomial Hopf algebra on one generator $\gamma_1$ and define a coaction $\psi\colon A \to \Gamma \otimes A$ by $\psi(X_n) = \sum_{i=0}^{n} \gamma_i \otimes X_{n-i}$. Then the primitive elements $P_\psi A$ form a polynomial algebra on generators in degrees $n\alpha$, $n \geq 2$. In §2 we construct specific polynomial generators for $P_\psi A$ as explicit polynomials in the $X_k$, $k \geq 1$.

In [2] we showed that this algebraic construction will construct polynomial generators for $H_*(BSU;R)$ in terms of the canonical polynomial generators $X_n = (C_1^n)^*$ of $H_*(BU;R)$ and will construct polynomial generators for $H_*(BSO;Z_2)$ in terms of the canonical generators $X_n = (W_1^n)^*$ of $H_*(BO;Z_2)$. In [2] polynomial generators were given for $H_*(BSU;Q)$, $H_*(BSU;Z_{(p)})$, $H_*(BSU;Z_p)$ and $H_*(BSO;Z_2)$. S. Papastavridis [3] has given polynomial generators for $H_*(BSO;Z_2)$. In §3 we compare these three sequences of polynomial generators for $H_*(BSO;Z_2)$.

We will use the notation $(a,b)$ for the binomial coefficient $\binom{a+b}{a}$. An elementary property of binomial coefficients which we exploit in our construction is

$$G_n = \mathrm{GCD}\{(i,n-i) \mid 1 \leq i \leq [n/2]\} = \begin{cases} p & \text{if } n \text{ is a power of a prime } p, \\ 1 & \text{otherwise.} \end{cases}$$

Thus there are integers $c,\ldots,c_{[n/2]}$ such that

$$\sum_{i=1}^{[n/2]} c_i(i,n-i) = G_n.$$

**2. Explicit polynomial generators for $P_\psi A$.** We fix an integer $n \geq 2$ and construct a polynomial generator $Y_{nn}$ of degree $n\alpha$ of $P_\psi A$ by successive approximations $Y_{n2},\ldots,Y_{nn}$. Our procedure will simultaneously construct a sequence of elements $Y_{kk} \in P_\psi A$ of degree $k\alpha$, $2 \leq k \leq n$, by successive approximations $Y_{k2},\ldots,Y_{kk}$. We obtain recursive relations for the $Y_{kk}$ and then solve them to obtain explicit

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formulas for the $Y_{kk}$ as polynomials in the $X_i$. Observe that throughout this section $Y_{ij}$ depend on $n$ and on a sequence $C$. We refrain, however, from denoting $Y_{ij}$ by $Y_{nij}(C)$. We begin by defining and studying elements $T(C)$ in $A$ which will be used to define our first approximations $Y_{22}, \ldots, Y_{n2}$.

DEFINITION 2.1. Let $C = (c_1, \ldots, c_{s-1})$, $s \geq 2$. Define

$$T(C) = \left[\sum_{i=1}^{s-1} c_i(i, s-i)\right] X_s - \sum_{i=1}^{s-1} c_i X_i X_{s-i}.$$

LEMMA 2.2. *Fix* $C = (c_1, \ldots, c_{n-1})$. *Then*

$$\psi(T(C)) = \sum_{i=0}^{n-2} \gamma_i \otimes T(C_{n-i})$$

*where* $C_{n-i} = (c_{n-i,1}, \ldots, c_{n-i,n-i-1})$ *and* $c_{n-i,k} = \sum_{h=0}^{i} c_{h+k}(h, i-h)$.

PROOF.

$$\begin{aligned}
\psi(T(C)) &= \left[\sum_{i=1}^{n-1} c_i(i, n-i)\right] \gamma_n \otimes 1 - \sum_{i=1}^{n-1} c_i \gamma_i \gamma_{n-i} \otimes 1 \\
&\quad + \left\{\sum_{i=1}^{n-1} c_i(i, n-i) - \sum_{i=1}^{n-1} c_i[(i-1, n-i) + (i, n-i-1)]\right\} \gamma_{n-1} \otimes X_1 \\
&\quad + \sum_{i=0}^{n-2} \gamma_i \otimes \left[\sum_{j=1}^{n-1} c_j(m, n-j) X_{n-j} \right. \\
&\qquad\qquad \left. - \sum_{j=1}^{n-1} \sum_{h=0}^{i} c_j(h, i-h) X_{j-h} X_{n-i-j+h}\right] \\
&\qquad\qquad\qquad \text{where } X_s = 0 \text{ for } s < 0, \\
&= \sum_{i=0}^{n-2} \gamma_i \otimes \left\{\left[\sum_{j=1}^{n-2} c_j(j, n-j)\right] X_{n-j} \right. \\
&\qquad\qquad \left. - \sum_{k=0}^{n-i} \left[\sum_{h=0}^{i} c_{h+k}(h, i-h)\right] X_k X_{n-i-k}\right\}
\end{aligned}$$

where $c_0 = c_n = 0$ and $k = j - h$. To identify the coefficient of $\gamma_i$ as $T(C_{n-i})$ we determine that the following difference is zero.

$$\begin{aligned}
&\sum_{j=1}^{n-1} c_j(j, n-j) - \sum_{k=0}^{n-i} \sum_{h=0}^{i} c_{h+k}(h, i-h)(k, n-i-k) \\
&\quad = \sum_{j=1}^{n-1} c_j(m, n-j) - \sum_{u=1}^{n-1} c_u \left[\sum_{h=0}^{i} \binom{i}{h} \binom{n-i}{u-h}\right] \quad \text{where } u = h+k \\
&\quad = \sum_{j=1}^{n-1} c_j(j, n-j) - \sum_{u=1}^{n-1} c_u \binom{n}{u} = 0.
\end{aligned}$$

LEMMA 2.3. *The $T(C_k)$, $2 \le k < n$, defined in Lemma 2.2 have coaction*

$$\psi(T(C_k)) = \sum_{i=0}^{k-2} \gamma_i \otimes T(C_{k-i}).$$

PROOF. This formula follows from Lemma 2.2 and the coassociativity of $\psi$.

LEMMA 2.4. *Fix $C = C_n = (c_1, \ldots, c_{n-1})$. Define $Y_{ki}$ for $2 \le i \le k \le n$ by induction on $i$ as follows.*

$$Y_{k2} = T(C_k) \quad \text{and}$$
$$Y_{ki} = Y_{k,i-1} - Y_{k-i+1}Y_{i-1,i-1} \quad \text{for } i \ge 3.$$

*Then $Y_{kk} \in P_\psi A$ for $2 \le k \le n$.*

PROOF. We prove the following formula for $2 \le i \le k \le n$ by induction on $i$. The case $i = k$ gives the desired conclusion.

$$(*) \qquad \psi(Y_{ki}) = \sum_{h=0}^{k-i} \gamma_h \otimes Y_{k-h,i}.$$

The case of $(*)$ when $i = 2$ is given by Lemmas 2.2 and 2.3. Assume that $i \ge 3$ and $(*)$ is true for the $Y_{k,i-1}$. Then

$$\begin{aligned}\psi(Y_{ki}) &= \psi(Y_{k,i-1} - X_{k-i+1}Y_{i-1,i-1}) \\ &= \sum_{h=0}^{k-i+1} \gamma_h \otimes Y_{k-h,i-1} - \sum_{h=0}^{k-i+1} \gamma_h \otimes X_{k-i-h+1}Y_{i-1,i-1} \\ &= \sum_{h=0}^{k-i} \gamma_h \otimes (Y_{k-h,i-1} - X_{k-i-h+1}Y_{i-1,i-1}) \\ &= \sum_{h=0}^{k-i} \gamma_h \otimes Y_{k-h,i}.\end{aligned}$$

THEOREM 2.5. *Choose $C = (c_1, \ldots, c_{n-1})$ such that $\sum_{i=1}^{n-1} c_i(i, n-i) = G_n$ in $R$. If $p = \operatorname{char} R$ is prime then assume that $n$ is not a power of $p$. Then the following element of $A$ is a polynomial generator of $P_\psi A$ in degree $n\alpha$.*

$$T(C) + \sum_{k=2}^{n-1} \overline{X}_{n-k} T(C_k).$$

*In this formula $T$ is given by Definition 2.1, $C_k$ is defined in Lemma 2.2 and*

$$\overline{X}_t = \sum_{e_1+2e_2+\cdots+te_t=t} (-1)^{e_1+\cdots+e_t}(e_1, \ldots, e_t) X_1^{e_1} \cdots X_t^{e_t}.$$

PROOF. By Lemma 2.4 we have

$$T(C_k) = Y_{kk} + \sum_{i=2}^{k-1} X_{k-i} Y_{ii} \qquad (2 \le k \le n).$$

Thus we have $n-1$ equations in the $n-1$ unknowns $Y_{22}, \ldots, Y_{nn}$. The coefficient matrix $M = (l_{ki})$ is lower triangular with ones on the diagonal. The $m_{ki}$ are in the commutative Hopf algebra $A$ whose diagonal $\Delta$ is defined by $\Delta(X_r) = \sum_{i=0}^{r} X_i \otimes X_{r-i}$. Note that

$$\Delta(m_{ki}) = \Delta(X_{k-i}) = \sum_{h=0}^{k-i} X_h \otimes X_{k-i-h} = \sum_{s=i}^{k} m_{ks} \otimes m_{si}.$$

Thus [1, Lemma 2.2] applies to solve this system of linear equations,

$$Y_{kk} = T(C_k) + \sum_{i=2}^{k-1} \overline{X}_{k-i} T(C_i).$$

The conjugates $\overline{X}_{k-i}$ are computed in [1, Theorem 4.1(v)]. Observe that $Y_{nn}$ is a polynomial generator by [2, Theorem 3.3].

*Note.* If char $R = p$ is prime then $X^p_{p^s}$, $s \geq 0$, are polynomial generators of $P_\psi A$ in degrees $\alpha p^s$, $s \geq 0$.

Let $A_0 = Z[X_1, \ldots, X_n, \ldots]$ and let $\eta\colon A_0 \to A$ be the canonical map induced by the unit $Z \to R$. The following easy result says that it suffices to study polynomial generators for $P_\psi A$ in the case $R = Z$. However, in practice it is often easier to study the case of interest directly. (See [2] and [3].)

PROPOSITION 2.6. *If* $P_\psi A_0 = Z[Y_2, \ldots, Y_k, \ldots]$ *then*

$$P_\psi A = R[\eta(Y_2), \ldots, \eta(Y_k), \ldots].$$

*Conversely, if* $P_\psi A_0 = R[Y'_2, \ldots, Y'_k, \ldots]$ *then there are* $Y_k$, $2 \leq k \leq n$, *in* $P_\psi A_0$ *such that* $P_\psi A_0 = Z[Y_2, \ldots, Y_k, \ldots]$ *and* $\eta(Y_k) = \mu_k Y'_k$ *for units* $\mu_2, \ldots, \mu_k, \ldots$ *in* $R$.

**3. Polynomial generators for $H_*(BSO; Z_2)$.** We begin by constructing an especially nice sequence of polynomial generators for $H_*(BSO; Z_2)$ of the type studied in §2. We then compare the three known sequences of polynomial generators for $H_*(BSO; Z_2)$ and find them to be distinct.

THEOREM 3.1. $H_*(BSO; Z_2) = Z_2[Q_2, \ldots, Q_n, \ldots]$ *where* $Q_{2^s} = X^2_{2^s-1}$, $s \geq 1$, *and if* $n = 2^s(2t+1)$, $t \geq 1$, *then*

$$Q_n = X_n + \sum_{i=1}^{t} X_{i2^s} X_{(2t+1-i)2^s} + \overline{X}_{2^s} X^2_{t2^s} + \sum_{k=1}^{2^s-1} \overline{X}_k T(C_{n-k}).$$

*The* $C_{n-k}$ *are determined by the sequence* $C = (c_1, \ldots, c_{n-1})$ *where*

$$c_1 = \begin{cases} 1 & \text{if } i = h2^s \text{ with } h \leq t, \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. $Q_n$ is the polynomial generator of degree $n$ determined by Theorem 2.5 from the sequence $C$.

Let $G_2, \ldots, G_n, \ldots$ be the sequence of polynomial generators for $H_*(BSO; Z_2)$ constructed in [2] and let $P_2, \ldots, P_n, \ldots$ be the sequence of polynomial generators for $H_*(BSO; Z_2)$ constructed by S. Papastavridis in [3]. The following theorem compares the $G_n$, $P_n$ and $Q_n$.

THEOREM 3.2. (a) $G_{2n+1} = P_{2n+1} = Q_{2n+1} = X_{2n+1} + \sum_{k=1}^{n} X_k X_{2n-k+1} + X_1 X_n^2$ *for* $n \geq 1$.

(b) $G_{2^n} = P_{2^n} = Q_{2^n} = X_{2^{n-1}}^2$ *for* $n \geq 1$.

(c) $P_{4n+2} = Q_{4n+2} + X_1^2 X_{2n}^2 = X_{4n+2} + \sum_{k=1}^{n} X_{2k} X_{4n-2k+2} + X_2 X_{2n}^2 + X_1 X_{4n+1} + \sum_{i=1}^{2n} X_1 X_i X_{4n-i+1}$ *for* $n \geq 1$.

(d) $G_{2^s(2t+1)}$, $P_{2^s(2t+1)}$ *and* $Q_{2^s(2t+1)}$ *are all distinct for* $t \geq 1$ *and* $s \geq 1$.

PROOF. (a), (b). The equality of the $G_i$, $P_i$ and $Q_i$ in these cases is clear from their definitions.

(c) $Q_{4n+2} = X_n + \sum_{i=1}^{n} X_{2i} X_{4n-2i+2} + \overline{X}_2 X_{2n}^2 + \overline{X}_1 T(C_{4n+1})$. Now $\overline{X}_2 = X_2 + X_1^2$, $\overline{X}_1 = X_1$ and $T(C_{4n+1}) = X_{4n+1} + \sum_{i=1}^{2n} X_i X_{4n-i+1}$. Thus

$$Q_{4n+2} = X_n + \sum_{i=1}^{n} X_{2i} X_{4n-i+2} + X_2 X_{2n}^2 + X_1 Q_{4n+1} = P_{4n+2} + X_1^2 X_{2n}^2.$$

(d) The polynomial generators $P_n$ constructed by S. Papastavridis have the distinctive property that they are in $H_*(BO(3); Z_2)$ i.e. they are in the $Z_2$-span of $\{X_k, X_i X_j, X_i X_j X_k \mid i, j, k \geq 1\}$. Clearly neither the $Q_n$ nor $P_n$ listed in (d) have this property. The $Q_n$ have the following property. Write $n = 2^s(2t+1)$ and $Q_n = \sum_{e_1 + \cdots + k e_k = n} \alpha_{e_1, \ldots, e_k} X_1^{e_1} \cdots X_k^{e_k}$. Then the coefficient of $X_i X_j$ in $Q_n$ is zero if $i \not\equiv 0 \bmod 2^s$ and $2^s \leq i \leq j$. The $G_n$ with $n = 2m$ not a power of two have the property that

$$G_n = X_n + \sum_{k=1}^{m} X_k X_{n-k} + \sum_{\substack{e_1 + \cdots + k e_k = n \\ e_1 + \cdots + e_k \geq 3}} \beta_{e_1, \ldots, e_k} X_1^{e_1} \cdots X_k^{e_k}.$$

Thus the $G_n$ and $Q_n$ listed in (d) are distinct.

## REFERENCES


1. S. O. Kochman, *Primitive generators for algebras*, Canad. J. Math. (to appear).
2. ——, *Polynomial generators for* $H_*(BSU)$ *and* $H_*(BSO; Z_2)$, Proc. Amer. Math. Soc. **84** (1982), 149–154.
3. S. Papastavridis, *The image of* $H_*(BSO; Z_2)$ *in* $H_*(BO; Z_2)$, preprint.


DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF WESTERN ONTARIO, LONDON, ONTARIO, CANADA N6A 5B7